

## ISOTROPIC SCHUR ROOTS

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**ABSTRACT.** In this paper, we study the isotropic Schur roots of an acyclic quiver  $Q$  with  $n$  vertices. We study the perpendicular category  $\mathcal{A}(d)$  of a dimension vector  $d$  and give a complete description of it when  $d$  is an isotropic Schur  $\delta$ . This is done by using exceptional sequences and by defining a subcategory  $\mathcal{R}(Q, \delta)$  attached to the pair  $(Q, \delta)$ . The latter category is always equivalent to the category of representations of a connected acyclic quiver  $Q_{\mathcal{R}}$  of tame type, having a unique isotropic Schur root, say  $\delta_{\mathcal{R}}$ . The understanding of the simple objects in  $\mathcal{A}(\delta)$  allows us to get a finite set of generators for the ring of semi-invariants  $\text{SI}(Q, \delta)$  of  $Q$  of dimension vector  $\delta$ . The relations among these generators come from the representation theory of the category  $\mathcal{R}(Q, \delta)$  and from a beautiful description of the cone of dimension vectors of  $\mathcal{A}(\delta)$ . Indeed, we show that  $\text{SI}(Q, \delta)$  is isomorphic to the ring of semi-invariants  $\text{SI}(Q_{\mathcal{R}}, \delta_{\mathcal{R}})$  to which we adjoin variables. In particular, using a result of Skowroński and Weyman, the ring  $\text{SI}(Q, \delta)$  is a polynomial ring or a hypersurface. Finally, we provide an algorithm for finding all isotropic Schur roots of  $Q$ . This is done by an action of the braid group  $B_{n-1}$  on some exceptional sequences. This action admits finitely many orbits, each such orbit corresponding to an isotropic Schur root of a tame full subquiver of  $Q$ .

## 1. INTRODUCTION

Let  $Q$  be an acyclic quiver with  $n$  vertices and  $k$  be an algebraically closed field. One crucial tool in representation theory of acyclic quivers is the use of perpendicular categories. If  $V$  is a rigid representation of  $Q$ , then the (left) perpendicular category  ${}^{\perp}V$  of  $V$  is an exact extension-closed abelian subcategory of  $\text{rep}(Q)$ , where  $\text{rep}(Q)$  is the category of finite dimensional representations of  $Q$ . These perpendicular subcategories were first studied by Geigle and Lenzing in [9] and also by Schofield in [18]. One important fact is that such a subcategory has a projective generator, or equivalently, it is equivalent to the category of representations of some acyclic quiver. There is a very natural way to generalize perpendicular categories of rigid representations, namely, by taking the perpendicular category  $\mathcal{A}(d)$  of a dimension vector  $d$ ; see Section 3. If  $V$  is a rigid representation, then  ${}^{\perp}V = \mathcal{A}(d_V)$  where  $d_V$  is the dimension vector of  $V$ . If  $d_V$  is not the dimension vector of a rigid representation, we show that the category  $\mathcal{A}(d)$  does not admit a projective generator. However, this category plays a fundamental role for understanding the ring  $\text{SI}(Q, d)$  of semi-invariants of  $Q$  of dimension vector  $d$ , the latter object being our second object of study in this paper. Indeed, as a ring,  $\text{SI}(Q, d)$  is generated by the Schofield semi-invariants  $C^V$  (see [6]) where  $V$  runs through the simple objects in  $\mathcal{A}(d)$ . In general,  $\mathcal{A}(d)$  may have infinitely many non-isomorphic simple objects. Since  $\text{SI}(Q, d)$  is always finitely generated as a ring, we still need to decide how to pick a nice subset of those generators  $C^V$  and find the relations among them. There is no general method, so far, for doing this. When we specialize to the case

where  $d = \delta$  is an isotropic Schur root of  $Q$ , then we can answer this problem. It was proven in [20] that when  $Q$  is of tame type and  $d$  is arbitrary, the ring of semi-invariants  $\text{SI}(Q, d)$  is either a polynomial ring or a hypersurface. The smallest possible case of hypersurface occurs for the dimension vector  $d = \delta$  where  $\delta$  is the isotropic Schur root for  $Q$  and where  $Q$  is of type  $\widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8$  or  $\widetilde{\mathbb{D}}_n$ , for  $n \geq 4$ . In such a case, the ring of semi-invariants is generated by the  $C^V$  where  $V$  runs through the quasi-simple exceptional regular representations of  $Q$ , that is, the exceptional simple objects in  $\mathcal{A}(\delta)$ . The hypersurface equation in these cases comes from the fact that the subcategory of regular representations in  $\text{rep}(Q)$  has exactly three non-homogeneous tubes. If the regular exceptional quasi-simple representations in the first, second and third tubes are respectively denoted  $E_1, \dots, E_p$ ,  $E'_1, \dots, E'_q$  and  $E''_1, \dots, E''_r$ , then the hypersurface equation is given by

$$C^{E_1} \dots C^{E_p} + C^{E'_1} \dots C^{E'_q} + C^{E''_1} \dots C^{E''_r} = 0.$$

We investigate how much of this structure is preserved for the ring  $\text{SI}(Q, \delta)$  of an isotropic Schur root  $\delta$  of an arbitrary  $Q$ . We consider the cone of dimension vectors in  $\mathcal{A}(\delta)$ . Using convex geometry and Radon's theorem, we prove a certain decomposition property of the space of that cone (Proposition 5.5). Using this, we give a complete description of the simple objects in  $\mathcal{A}(\delta)$ . There exists an exact extension-closed abelian subcategory  $\mathcal{R} = \mathcal{R}(Q, \delta)$  of  $\text{rep}(Q)$  which has a projective generator, connected and of tame type. In particular,  $\mathcal{R}$  is equivalent to  $\text{rep}(Q_{\mathcal{R}})$  for a connected quiver of tame type  $Q_{\mathcal{R}}$  having a unique isotropic Schur root  $\delta_{\mathcal{R}}$ , that can also be seen as an isotropic Schur root of  $Q$ . This subcategory  $\mathcal{R}$  is built from the data of an exceptional sequence  $(M_{n-2}, \dots, M_1)$  of length  $n-2$  of simple objects in  $\mathcal{A}(\delta)$ . Up to isomorphisms, the simple objects in  $\mathcal{A}(\delta)$  are given by the objects  $M_{n-2}, \dots, M_1$  together with the quasi-simple objects in  $\mathcal{R}$ . In particular, there are finitely many, up to isomorphism, exceptional simple objects in  $\mathcal{A}(\delta)$ . Using this, we can get an explicit description of our ring of semi-invariants:  $\text{SI}(Q, \delta)$  is obtained by adjoining variables to the ring of semi-invariants  $\text{SI}(Q_{\mathcal{R}}, \delta_{\mathcal{R}})$ . In particular,  $\text{SI}(Q, \delta)$  is still a polynomial ring or a hypersurface. The defining equation, in the hypersurface case, again comes from linear dependance of three products of semi-invariants  $C^V$  where  $V$  runs through the exceptional quasi-simple objects in  $\mathcal{R}(Q, \delta)$ . One difference between the general case and the case of quivers of tame type is that the isotropic Schur root  $\delta_{\mathcal{R}}$  may differ from  $\delta$ . Moreover, the root  $\delta_{\mathcal{R}}$  needs not lie in the interior of the cone of dimension vectors of  $\mathcal{A}(\delta)$ . For the last part of this paper, we find an algorithm for finding all isotropic Schur roots of  $Q$ . We restrict to full exceptional sequences of  $\text{rep}(Q)$  that are called exceptional of isotropic type: it is an exceptional sequence of the form  $E = (X_1, \dots, X_{n-1}, X_n)$  where there is an integer  $i$  such that the thick subcategory generated by  $X_i, X_{i+1}$  contains an isotropic Schur root  $\delta_E$ , called the type of  $E$ . We explain how the braid group  $B_{n-1}$  acts on these sequences to get all the isotropic Schur roots of  $Q$ . The action admits finitely many orbits, and each orbit contains an exceptional sequence whose type is an isotropic Schur root of a tame full subquiver of  $Q$ .

## 2. PRELIMINARIES

In this paper,  $Q = (Q_0, Q_1)$  is always a connected acyclic quiver with  $n$  vertices, unless otherwise indicated, and  $k$  is an algebraically closed field. We let  $\text{rep}(Q)$  denote the category of ( $k$ -linear) finite dimensional representations of  $Q$  over  $k$ .

**2.1. Bilinear form, Schur roots.** Given a representation  $M$ , we denote by  $d_M$  its dimension vector, which is an element in  $(\mathbb{Z}_{\geq 0})^n$ . We denote by  $\langle -, - \rangle_Q$ , or simply by  $\langle -, - \rangle$  when there is no risk of confusion, the *Euler-Ringel* form for  $Q$ . This is the unique  $k$ -bilinear form in  $\mathbb{R}^n$  such that for  $M, N \in \text{rep}(Q)$ , we have

$$\langle d_M, d_N \rangle = \dim_k \text{Hom}(M, N) - \dim_k \text{Ext}^1(M, N).$$

Recall that  $M \in \text{rep}(Q)$  is called a *Schur representation* if  $\text{Hom}(M, M)$  is one dimensional. It is well known from Kac [12, Lemma 2.1] that in this case,  $\langle d_M, d_M \rangle$  is at most one. A Schur representation  $M$  is called an *exceptional representation* if  $\langle d_M, d_M \rangle = 1$  or, equivalently,  $\text{Ext}^1(M, M) = 0$ . A dimension vector  $d \in (\mathbb{Z}_{\geq 0})^n$  is a *Schur root* if  $d = d_M$  for some Schur representation  $M$ . We call such a  $d$  *real*, *isotropic* or *imaginary* if  $\langle d, d \rangle$  is one, zero or negative, respectively. In this paper,  $\delta$  will always denote an isotropic Schur root of  $Q$ .

**2.2. Geometry and semi-invariants.** For an arrow  $\alpha \in Q_1$ , we denote by  $t(\alpha)$  its *tail* and by  $h(\alpha)$  its *head*. We write  $\text{Mat}_{u \times v}(k)$  for the set of all  $u \times v$  matrices over  $k$ . For a dimension vector  $d = (d_1, \dots, d_n)$ , we denote by  $\text{rep}(Q, d)$  the space of representations of dimension vector  $d$  with fixed vector spaces, that is,

$$\text{rep}(Q, d) = \prod_{\alpha \in Q_1} \text{Mat}_{t(\alpha) \times h(\alpha)}(k).$$

This space is an affine space and the reductive group  $\text{GL}_d(k) := \prod_{i=1}^n \text{GL}_{d_i}(k)$  acts on it by simultaneous conjugation, so that for  $M \in \text{rep}(Q, d)$ , the  $\text{GL}_d(k)$ -orbit of  $M$  is the set of all representations in  $\text{rep}(Q, d)$  that are isomorphic to  $M$ . Since  $Q$  is acyclic, the ring of invariants  $k[\text{rep}(Q, d)]^{\text{GL}_d(k)}$  is trivial. Instead, one rather considers the ring of invariants  $k[\text{rep}(Q, d)]^{\text{SL}_d(k)}$ , where

$$\text{SL}_d(k) := \prod_{i=1}^n \text{SL}_{d_i}(k) \subset \text{GL}_d(k).$$

This ring, denoted  $\text{SI}(Q, d)$ , is called the *ring of semi-invariants* of  $Q$  of dimension vector  $d$ . It is always finitely generated, since  $\text{SL}_d(k)$  is reductive. Let  $\Gamma$  denote the group of all homomorphisms  $\mathbb{Z}^n \rightarrow \mathbb{Z}$ , which is identified to the set of multiplicative characters of  $\text{GL}_d(k)$ . A well known result states that  $\text{SI}(Q, d)$  admits a weight space decomposition

$$\text{SI}(Q, d) = \bigoplus_{\tau \in \Gamma} \text{SI}(Q, d)_{\tau}$$

where  $\text{SI}(Q, d)_{\tau}$  is called the space of *semi-invariants of weight  $\tau$* . This makes  $\text{SI}(Q, d)$  a  $\Gamma$ -graded ring.

If  $V$  is a representation with  $\langle d_V, d \rangle = 0$  and  $f_V : P_1 \rightarrow P_0$  denotes a fixed projective resolution of  $V$ , then for  $M \in \text{rep}(Q, d)$ , we have a  $k$ -linear map  $d(V, M) = \text{Hom}(f_V, M)$  given by a square matrix. We define  $C^V(-)$  to be the polynomial function on  $\text{rep}(Q, d)$  that takes a representation  $M$  to the determinant of  $d(V, M)$ . If we change the projective resolution of  $V$ ,  $C^V(-)$  only changes by a non-zero scalar. This function  $C^V(-)$  is a semi-invariant of weight  $\langle d_V, - \rangle$ , and will be called *determinantal semi-invariant*; see [6, 18]. The following theorem was proven by Derksen and Weyman in [6] and also by Schofield and van den Bergh in [19] in the characteristic zero case.

**Theorem 2.1** (Derksen-Weyman, Schofield-Van den Bergh). *Let  $d$  be a dimension vector. Then the ring  $\text{SI}(Q, d)$  is spanned over  $k$  by the determinantal semi-invariants  $C^V(-)$  where  $\langle d_V, d \rangle = 0$ .*

There is a dual way to define semi-invariants. One can also take a representation  $W$  with  $\langle d, d_W \rangle = 0$  and  $f_W : P'_1 \rightarrow P'_0$  a fixed projective resolution of  $W$ . For  $M \in \text{rep}(Q, d)$ , we have a  $k$ -linear map  $d(M, W) = \text{Hom}(M, f_W)$  given by a square matrix. We define  $C^-(W)$  the polynomial function on  $\text{rep}(Q, d)$  that takes a representation  $M$  to the determinant of  $d(M, W)$ . This function  $C^-(W)$  is a semi-invariant of weight  $-\langle -, d_W \rangle$ . As in the above theorem, the ring  $\text{SI}(Q, d)$  is spanned over  $k$  by the semi-invariants  $C^-(W)$  where  $\langle d, d_W \rangle = 0$ .

**2.3. Exceptional sequences, thick subcategories and braid groups.** One of our main tools in this paper will be to make use of particular exceptional sequences. Recall that a sequence

$$(X_1, \dots, X_r)$$

of exceptional representations is an *exceptional sequence* if

$$\text{Hom}(X_i, X_j) = 0 = \text{Ext}^1(X_i, X_j)$$

whenever  $i < j$ . It is *full* if  $r = n$ . For such an exceptional sequence  $E$ , we denote by  $\mathcal{C}(E)$  the thick subcategory of  $\text{rep}(Q)$  generated by the objects in  $E$ . Here *thick* means full, closed under extensions, under direct summands, under kernels of epimorphisms, and under cokernels of monomorphisms. The following is well known. We include a proof for the sake of completeness.

**Proposition 2.2.** *A full subcategory  $\mathcal{A}$  of  $\text{rep}(Q)$  is thick if and only if it is exact abelian and extension-closed.*

*Proof.* The sufficiency is clear. Suppose that  $\mathcal{A}$  is thick. We only need to show that  $\mathcal{A}$  has kernels and cokernels. We will prove this by showing that the kernel and cokernel in  $\text{rep}(Q)$  of a morphism in  $\mathcal{A}$  lie in  $\mathcal{A}$ . Let  $f : M \rightarrow N$  be a morphism in  $\mathcal{A}$  and let  $u : K \rightarrow M$ ,  $v : N \rightarrow C$  and  $g : M \rightarrow E$  be the kernel, cokernel and coimage in  $\text{rep}(Q)$ . Since  $\text{rep}(Q)$  is hereditary and since we have a monomorphism  $g' : E \cong \text{im}(f) \rightarrow N$ , we have a surjective map  $\text{Ext}^1(g', K) : \text{Ext}^1(N, K) \rightarrow \text{Ext}^1(E, K)$ . The short exact sequence  $0 \rightarrow K \rightarrow M \rightarrow E \rightarrow 0$  is an element in  $\text{Ext}^1(E, K)$  and hence is the image of an element in  $\text{Ext}^1(N, K)$ . We have a pullback diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \xrightarrow{u} & M & \xrightarrow{g} & E \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow g' \\ 0 & \longrightarrow & K & \longrightarrow & M' & \longrightarrow & N \longrightarrow 0 \end{array}$$

This gives rise to a short exact sequence

$$0 \rightarrow M \rightarrow M' \oplus E \rightarrow N \rightarrow 0.$$

Since  $M, N \in \mathcal{A}$  and  $\mathcal{A}$  is thick, we get  $E \in \mathcal{A}$ . Hence,  $K, C \in \mathcal{A}$ .  $\square$

Recall that if  $E$  is an exceptional sequence of length  $r \leq n$ , then  $\mathcal{C}(E)$  is equivalent to the category  $\text{rep}(Q_E)$  of representations of an acyclic quiver  $Q_E$  with  $r$  vertices; see [18]. Denote by  $S_1, \dots, S_r$  the non-isomorphic simple objects in  $\mathcal{C}(E)$ . The Euler-Ringel form of  $\text{rep}(Q)$ , restricted to the subgroup of  $\mathbb{Z}^n$  generated by the  $d_{S_i}$ , is isometric to the Euler-Ringel form of  $\text{rep}(Q_E)$ . In other words, there is

an equivalence  $\psi : \text{rep}(Q_E) \rightarrow \mathcal{C}(E)$  of categories such that for  $X, Y \in \text{rep}(Q_E)$ , we have

$$\langle d_{\psi(X)}, d_{\psi(Y)} \rangle_Q = \langle d_X, d_Y \rangle_{Q_E}.$$

In this way, we will often identify a dimension vector for  $Q_E$  to a dimension vector for  $Q$ . Moreover, using the above equivalence, a Schur root for  $Q_E$  will be thought of as a Schur root for  $Q$ .

Let  $E = (X_1, \dots, X_r)$  be an exceptional sequence. If  $j > 1$ , we denote by  $L_{X_{j-1}}(X_j)$  the reflection of  $X_j$  to the left of  $X_{j-1}$ . This is the unique exceptional representation such that we have an exceptional sequence

$$(X_1, \dots, L_{X_{j-1}}(X_j), X_{j-1}, X_{j+1}, \dots, X_r).$$

If  $j < r$ , we denote by  $R_{X_{j+1}}(X_j)$  the reflection of  $X_j$  to the right of  $X_{j+1}$ . This is the unique exceptional representation such that we have an exceptional sequence

$$(X_1, \dots, X_{j-1}, X_{j+1}, R_{X_{j+1}}(X_j), X_{j+2}, \dots, X_r).$$

These reflections actually have another interpretation in terms of the braid group. Let  $r = n$ . For  $1 \leq i \leq n-1$ , denote by  $\sigma_i$  the operation that takes an exceptional sequence  $E$  and reflect the  $(i+1)$ th object to the left of the  $i$ th one. Hence,  $\sigma_1, \dots, \sigma_{n-1}$  act on the set of all exceptional sequences. It is not hard to check that for  $1 \leq i \leq n-2$  and an exceptional sequence  $E$ , we have

$$(\sigma_i(\sigma_{i+1}(\sigma_i E))) = (\sigma_{i+1}(\sigma_i(\sigma_{i+1} E)))$$

and for  $|i-j| \geq 2$ , we have

$$(\sigma_i(\sigma_j E)) = (\sigma_i(\sigma_j E)).$$

Therefore, the braid group

$$B_n := \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| \geq 2 \rangle$$

on  $n$  strands acts on the set of all full exceptional sequences. It is well known by a result of Crawley-Boevey [3] that this action is transitive, meaning that all exceptional sequences lie in a single orbit.

### 3. PERPENDICULAR SUBCATEGORIES, STABLE AND SEMI-STABLE REPRESENTATIONS

In this section, we consider perpendicular subcategories of dimension vectors, study them and explain why these categories are related to semi-invariants of quivers. Let  $d$  be any dimension vector. Consider

$$\mathcal{A}(d) := \{X \in \text{rep}(Q) \mid \text{Hom}(X, M) = 0 = \text{Ext}^1(X, M) \text{ for some } M \in \text{rep}(Q, d)\},$$

seen as a full subcategory. This category is called the *(left) perpendicular subcategory* of  $d$ .

**Proposition 3.1.** *The subcategory  $\mathcal{A}(d)$  is a thick subcategory of  $\text{rep}(Q)$ , hence exact extension-closed abelian.*

*Proof.* It is clear that  $\mathcal{A}(d)$  is closed under direct summands. Let  $0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow 0$  be a short exact sequence in  $\text{rep}(Q)$  with exactly two terms  $X_s, X_t$  in  $\mathcal{A}(d)$ . There are  $N_s, N_t \in \text{rep}(Q, d)$  such that  $\text{Hom}(X_i, N_i) = 0 = \text{Ext}^1(X_i, N_i)$  whenever  $i = s, t$ . Now, for  $i = s, t$ , there is an open set  $\mathcal{U}_i \in \text{rep}(Q, d)$  such that for  $Z_i \in \mathcal{U}_i$ , we have  $\text{Hom}(X_i, Z_i) = 0 = \text{Ext}^1(X_i, Z_i)$ . Since  $\mathcal{U}_s \cap \mathcal{U}_t$  is non-empty, take  $N$  a

representation in  $\mathcal{U}_s \cap \mathcal{U}_t$ . Applying  $\text{Hom}(-, N)$  to the above exact sequence, we get that the third term lies in  $\mathcal{A}(d)$ .  $\square$

Clearly,  $\mathcal{A}$  has a projective generator if and only if it is equivalent to a module category. Since  $\mathcal{A}$  is a Hom-finite hereditary abelian category over an algebraically closed field, this means that  $\mathcal{A}$  has a projective generator if and only if it is equivalent to  $\text{rep}(Q')$  for some finite acyclic quiver  $Q'$ .

**Lemma 3.2.** *Let  $d$  be an imaginary or isotropic Schur root in  $\text{rep}(Q)$  with  $n = 2$ . Then  $\mathcal{A}(d)$  is not empty.*

*Proof.* The result is trivial if  $d$  is isotropic. Therefore, assume that the quiver contains at least  $m \geq 3$  arrows and  $d$  is imaginary. Assume also that the vertices are  $\{1, 2\}$  with 1 being the sink vertex. It is sufficient to prove that the result holds for some dimension vector  $f$  in the  $\tau$ -orbit of  $d$ , where  $\tau$  denotes the Coxeter transformation. Consider the cone  $C = \{x \in (\mathbb{Z}_{\geq 0})^2 \mid \langle x, x \rangle < 0\}$ . The Coxeter transformation is clearly  $C$ -invariant. Observe that  $\tau$  sends the vector  $[m-1, 1]$  to  $[1, m-1]$ . Set  $z = [m-1, 1]$  and for  $i \in \mathbb{Z}$ , set  $z_i = \tau^i z$ . We claim that for  $i \in \mathbb{Z}$ , the cones  $[z_i, z_{i+1}]$ ,  $[z_{i+1}, z_{i+2}]$  only intersect at the ray generated by  $z_{i+1}$ . Assume otherwise. By continuity of  $\tau$  and since  $\tau z_i = z_{i+1}$ , there exists a ray in  $[z_i, z_{i+1}]$  that is fixed by  $\tau$ . Therefore, there is an eigenvector  $v$  in  $[z_i, z_{i+1}]$ . Then  $\langle v, v \rangle = -\langle v, \tau v \rangle = -\lambda \langle v, v \rangle$  where  $\lambda$  is the corresponding eigenvalue. It is well known, see [16], that  $\tau$  has a real eigenvalue greater than 1 and a positive real eigenvalue less than one. Therefore,  $\lambda \neq -1$  meaning that  $\langle v, v \rangle = 0$  so  $v \notin C$ , a contradiction. This proves our claim. This also proves that the two linearly independent eigenvectors of  $\tau$  lie on the two boundary rays of  $C$ . Now, the vectors  $z_1, z_2, z_3, \dots$  converge to one such ray and  $z_0, z_{-1}, z_{-2}, \dots$  converge to the other ray. Therefore, we see that the dimension vectors in the half open cone  $[z_0, z_1)$  forms a fundamental domain in  $C$  for the action of  $\tau$ . Hence, we may assume that our dimension vector  $d$  lies in  $[z_0, z_1]$ .

Assume first that  $d = [q, p]$  with  $1 \leq p \leq q$ . Since  $d$  lies in  $[z_0, z_1]$ , we have  $1 \leq \frac{q}{p} \leq m-1$ . By [2, Theo. 1], the Hilbert null-cone of  $\text{rep}(Q, d)$  is not the entire space if  $m > \lceil \frac{q}{p} \rceil$  and  $p > 1$ . The first condition always holds since  $d \in [z_0, z_1]$ . So if  $p > 1$ , then there has to be a representation  $M$  of dimension vector  $d$  and a semi-invariant that does not vanish at  $M$ . By Theorem 2.1, this yields a representation  $N$  with  $C^N(M) \neq 0$ , meaning that  $N \in \mathcal{A}(d)$ . Assume now that  $p = 1$  and take  $M$  a general representation of dimension vector  $d$ . Consider a general representation  $Z$  of dimension vector  $[mq-1, q]$ . By construction  $\langle d_Z, d_M \rangle = 0$ . Now, a proper subrepresentation of  $M$  has dimension vector  $d_i = [i, 0]$  for  $0 \leq i \leq q$ . Notice that  $\langle d_Z, d_i \rangle \leq 0$ . Therefore, it follows from King's criterion [14] that  $M$  is  $\langle d_Z, - \rangle$ -semistable. Therefore, there is a positive integer  $r$  and a representation  $Z'$  of dimension vector  $rd_Z$  such that  $C^{Z'}(M) \neq 0$ , meaning that  $Z' \in \mathcal{A}(d)$ . Similarly, one can prove that  $\mathcal{A}(d)$  contains a non-zero representation if  $d = [q, p]$  with  $1 \leq q \leq p$ .  $\square$

For two dimension vectors  $d_1, d_2$ , let  $\text{ext}(d_1, d_2)$  denote the minimal value of  $\dim_k \text{Ext}^1(M_1, M_2)$  where  $(M_1, M_2) \in \text{rep}(Q, d_1) \times \text{rep}(Q, d_2)$ . Similarly, we let  $\text{hom}(d_1, d_2)$  denote the minimal value of  $\dim_k \text{Hom}(M_1, M_2)$  where  $(M_1, M_2) \in \text{rep}(Q, d_1) \times \text{rep}(Q, d_2)$ . There is an open subset  $\mathcal{U}_1$  of  $\text{rep}(Q, d_1)$  and an open

subset  $\mathcal{U}_2$  of  $\text{rep}(Q, d_2)$  such that for  $M_1 \in \mathcal{U}_1, M_2 \in \mathcal{U}_2$ , we have

$$\langle d_1, d_2 \rangle = \dim_k \text{Hom}(M_1, M_2) - \dim_k \text{Ext}^1(M_1, M_2) = \text{hom}(d_1, d_2) - \text{ext}(d_1, d_2).$$

We write  $d_1 \perp d_2$  if  $\text{hom}(d_1, d_2) = \text{ext}(d_1, d_2) = 0$ . Observe that  $d_1 \perp d_2$  implies  $\langle d_1, d_2 \rangle = 0$ . A sequence  $(d_1, \dots, d_r)$  of Schur roots with  $d_i \perp d_j$  whenever  $i < j$  is called an *orthogonal sequence of Schur roots*.

Let  $d$  be a dimension vector. Due to results of Kac [13], there is a decomposition, denoted  $d = \alpha_1 \oplus \dots \oplus \alpha_m$ , having the property that there exists an open (dense) subset  $\mathcal{U}_d$  of  $\text{rep}(Q, d)$  such that for  $M \in \mathcal{U}_d$ , we have a decomposition  $M \cong M_1 \oplus \dots \oplus M_m$ , where each  $M_i$  is a Schur representation with  $d_{M_i} = \alpha_i$ . Moreover,  $\text{Ext}^1(M_i, M_j) = 0$  when  $i \neq j$ . The latter decomposition of  $d$  is unique up to ordering, and is called the *canonical decomposition* of  $d$ . The dimension vectors  $\alpha_i$  are clearly Schur roots, however, they do not need be distinct. Sometimes, it is more convenient to write the above decomposition as

$$(*) \quad d = p_1 \beta_1 \oplus \dots \oplus p_r \beta_r,$$

where the  $\beta_i$  are pairwise distinct and  $p_i$  is the number of  $1 \leq j \leq m$  with  $\beta_i = \alpha_j$ . It follows from [17, Theorem 3.8] that when  $\beta_i$  is imaginary, then  $p_i = 1$ . When writing a canonical decomposition as in  $(*)$ , we adopt the convention that when  $\alpha$  is an imaginary Schur root and  $p$  is a positive integer, then  $p\alpha$  is just one root (not  $p$  times the root  $\alpha$  as when  $\alpha$  is real or isotropic). With this convention, Schofield has proven in [17, Theorem 3.8] the following result.

**Proposition 3.3** (Schofield). *Let  $d = p_1 \beta_1 \oplus \dots \oplus p_r \beta_r$  be the canonical decomposition of  $d$ . If  $p$  is a positive integer, then  $pd = pp_1 \beta_1 \oplus \dots \oplus pp_r \beta_r$  is the canonical decomposition of  $pd$ , using the above convention for imaginary Schur roots.*

The following generalizes Lemma 3.2.

**Lemma 3.4.** *Let  $d$  be a dimension vector in  $\text{rep}(Q)$  whose canonical decomposition involves an isotropic Schur root or an imaginary Schur root. Then  $\mathcal{A}(d)$  is not empty.*

*Proof.* We follow the algorithm in [4] to find the canonical decomposition of  $d$ . In particular, we apply this algorithm until the first step where an imaginary or isotropic Schur root is created. In particular, there is an orthogonal sequence  $(\alpha_1, \dots, \alpha_r)$  of real Schur roots with positive integers  $p_1, \dots, p_r$  such that  $d = \sum_{i=1}^r p_i \alpha_i$ . Moreover, there is a pair  $\alpha_t, \alpha_{t+1}$  with  $\langle \alpha_{t+1}, \alpha_t \rangle < 0$  and  $\gamma = p_t \alpha_t + p_{t+1} \alpha_{t+1}$  is such that  $\langle \gamma, \gamma \rangle \leq 0$ .

Assume first that  $r < n$ . Then we can find a real Schur root  $\alpha_0$  such that  $\alpha_0 \perp \alpha_j$  for  $1 \leq j \leq r$ . Then, an exceptional representation of dimension vector  $\alpha_0$  lies in  $\mathcal{A}(d)$ . Therefore, we may assume that  $r = n$ . Let  $\gamma' = \gamma$  if  $\langle \gamma, \gamma \rangle < 0$  and  $\gamma'$  be the smallest indivisible dimension vector in the ray of  $\gamma$ , otherwise. Set  $p \geq 1$  with  $\gamma = p\gamma'$ . Observe that  $\gamma'$  is a Schur root. The next step of the algorithm replaces the sequence  $(\alpha_1, \dots, \alpha_n)$  with positive integers  $p_i$  by the orthogonal sequence  $(\alpha_1, \dots, \alpha_{t-1}, \gamma', \alpha_{t+2}, \dots, \alpha_n)$  of Schur roots with positive integers  $p_1, \dots, p_{t-1}, p, p_{t+2}, \dots, p_n$ . Now,

$$d = \sum_{i=1}^{t-1} p_i \alpha_i + p\gamma' + \sum_{i=t+2}^n p_i \alpha_i.$$

For  $1 \leq i \leq n$ , let  $M_i$  denote an exceptional representation of dimension vector  $\alpha_i$ . The root  $\gamma'$  is a root in  $\mathcal{C}(M_t, M_{t+1})$ . Since  $\mathcal{C}(M_t, M_{t+1})$  is equivalent to the category of representations of a quiver with two vertices, it follows from Lemma 3.2 that there is a dimension vector  $\nu$  in  $\mathcal{C}(M_t, M_{t+1})$  with  $\nu \perp \gamma'$ . Clearly,  $\langle \nu, \nu \rangle < 0$  and hence  $\nu$  is an imaginary Schur root. Let  $Z$  be a general representation of dimension vector  $\nu$ . Since  $Z$  is in general position, we may assume that it lies in  $M_j^\perp$  for  $1 \leq j \leq t-1$  and in  ${}^\perp M_j$  for  $t+2 \leq j \leq n$ . Therefore, we may assume it lies in  $\mathcal{C}(M_t, M_{t+1})$ . Since  $Z, M_{t-1}$  are in general position and  $\text{Ext}^1(M_{t-1}, Z) = 0$ , either  $\text{Hom}(Z, M_{t-1}) = 0$  or  $\text{Ext}^1(Z, M_{t-1}) = 0$  by [17, Theo. 4.1]. We construct a representation  $Z'$  that lies in  $M_j^\perp$  for  $1 \leq j \leq t-2$ , in  ${}^\perp M_j$  for  $j = t-1$  and  $t+2 \leq j \leq n$  and lies in  ${}^\perp N$  where  $N$  is a general representation of dimension vector  $\gamma'$ .

Assume first that  $\text{Hom}(Z, M_{t-1}) = 0$ . It follows from [4, Cor. 14] that  $\nu' = \nu - \langle \nu, \alpha_{t-1} \rangle \alpha_{t-1}$  is a Schur root with  $\text{hom}(\nu', \alpha_{t-1}) = 0$ . Moreover, since  $\langle \nu', \alpha_{t-1} \rangle = 0$ , we also have  $\text{ext}^1(\nu', \alpha_{t-1}) = 0$ . Therefore, we have an orthogonal sequence

$$(\alpha_1, \dots, \alpha_{t-2}, \nu', \alpha_{t-1}, \gamma', \alpha_{t+2}, \dots, \alpha_n)$$

of Schur roots. Consider  $Z'$  a general representation of dimension vector  $\nu'$ . Then  $Z'$  satisfies the above wanted conditions for  $Z'$ .

Assume now that  $\text{Ext}^1(Z, M_{t-1}) = 0$ . It follows from Lemma 2.3 in [17] that the non-zero morphisms  $Z \rightarrow M_{t-1}$  are either all injective or all surjective. Let  $f = \dim_k \text{Hom}(Z, M_{t-1})$ . Assume first that all non-zero morphisms  $Z \rightarrow M_{t-1}$  are surjective. We get an epimorphism  $Z \rightarrow M_{t-1}^f$  given by the basis elements of  $\text{Hom}(Z, M_{t-1})$ . It is not hard to check that the kernel  $Z'$  lies in  ${}^\perp M_{t-1}$ . Clearly,  $Z'$  satisfies the above wanted conditions for  $Z'$ . Similarly, we can construct such a  $Z'$  if the non-zero morphisms  $Z \rightarrow M_{t-1}$  are injective. We can continue this process and construct a representation  $X$  that lies in  $M_j^\perp$  for  $1 \leq j \leq t-1$  and  $t+2 \leq j \leq n$  and lies in  ${}^\perp N$  where  $N$  is a general representation of dimension vector  $\gamma'$ . In particular,  $X$  lies in  $\mathcal{A}(d)$ . Observe that  $X$  is not the zero representation since  $\alpha_1, \dots, \alpha_{t-1}, \nu, \gamma'$  are linearly independent.  $\square$

Recall that a representation  $V \in \text{rep}(Q)$  is *rigid* if  $\text{Ext}^1(V, V) = 0$ . Hence, the indecomposable rigid representations are the exceptional ones. Observe that there is a one-to-one correspondence

$$\{\text{real Schur roots}\} \leftrightarrow \{\text{iso. classes of exceptional representations}\}.$$

The dimension vector of a rigid representation is called *prehomogeneous*. Observe that if  $V$  is rigid, then the  $\text{GL}(d_V)$ -orbit of  $V$  is open in  $\text{rep}(Q, d_V)$ . In this case, the canonical decomposition of  $d$  involves the dimension vectors of its indecomposable direct summands, so involves only real Schur roots. Conversely, if the canonical decomposition of a dimension vector  $d$  involves only real Schur roots, then  $\text{rep}(Q, d)$  has an open orbit and hence,  $d = d_V$  where  $V$  is rigid. Thus, the above correspondence extends to the following one-to-one correspondence:

$$\{d \mid d \text{ is prehomogeneous}\} \leftrightarrow \{\text{iso. classes of rigid representations}\}.$$

**Proposition 3.5.** *The subcategory  $\mathcal{A}(d)$  has a projective generator if and only if  $d$  is prehomogeneous. In this case,*

$$\mathcal{A}(d) = {}^\perp V = \{X \in \text{rep}(Q) \mid \text{Hom}(X, V) = 0 = \text{Ext}^1(X, V)\}$$

where  $V$  is rigid with  $d = d_V$ .



*Proof.* Assume that  $\mathcal{A}(d)$  has a projective generator. Therefore, it is equivalent to the category of representations of an acyclic quiver. Let  $M_1, \dots, M_r$  be the indecomposable simple objects in  $\mathcal{A}(d)$ , up to isomorphism. We may assume that they are ordered so that  $E := (M_1, \dots, M_r)$  is an exceptional sequence in  $\mathcal{A}(d)$  and hence, also an exceptional sequence in  $\text{rep}(Q)$ . We have  $\mathcal{A}(d) = \mathcal{C}(E)$ . As argued in the proof of Proposition 3.1, for  $1 \leq i \leq r$ , there are open sets  $\mathcal{U}_i$  in  $\text{rep}(Q, d)$  such that for  $N_i \in \mathcal{U}_i$ , we have  $\text{Hom}(M_i, N_i) = 0 = \text{Ext}^1(M_i, N_i)$ . Now,  $\bigcap \mathcal{U}_i$  is non-empty and we let  $N$  lie in it. Then, for  $1 \leq i \leq r$ , we have  $\text{Hom}(M_i, N) = 0 = \text{Ext}^1(M_i, N)$ . Since the  $M_i$  are the simple objects in  $\mathcal{C}$ , it follows that  $N \in \mathcal{C}(E)^\perp$ . Thus,  $\mathcal{A}(d) \subseteq {}^\perp N$ . However, by definition of  $\mathcal{A}(d)$ , we have  ${}^\perp N \subseteq \mathcal{A}(d)$ . Therefore,  $\mathcal{A}(d) = {}^\perp N$ . Observe that  $\mathcal{C}(E)^\perp$  is equivalent to the category of representations of an acyclic quiver  $Q'$ . We can think of  $N$  as a representation in  $\text{rep}(Q')$  with dimension vector  $d$ . Assume that  $N$  is not rigid. Then the canonical decomposition of  $d$  (as a dimension vector of  $Q'$ ) involves an isotropic or imaginary Schur root of  $Q'$ . It follows from Lemma 3.4 that there is a representation  $Z$  in the category  $\mathcal{A}(d)$  for  $\text{rep}(Q')$ . This means that  $Z \in \mathcal{A}(d)$  but  $Z \notin \mathcal{C}(E)$ , a contradiction. Therefore,  $N$  is rigid and this proves the necessity. Assume now that  $d = d_V$  where  $V$  is rigid. Since  $V$  is in general position, any  $Z \in \mathcal{A}(d)$  has to be in  ${}^\perp V$  and hence,  $\mathcal{A}(d) = {}^\perp V$ .  $\square$

Given a dimension vector  $d$ , we fix  $\sigma_d := -\langle -, d \rangle$ , which is called the *weight associated to  $d$* . Recall that  $M \in \text{rep}(Q)$  is  $\sigma_d$ -semistable if there is a positive integer  $m$  and a semi-invariant  $f$  of weight  $m\sigma_d$  in  $\text{SI}(Q, d_M)$  such that  $f$  does not vanish at  $M$ . If  $M$  is  $\sigma_d$ -semistable and has no proper (non-zero)  $\sigma_d$ -semistable subobject, then it is  $\sigma_d$ -stable. It follows from King's criterion [14] that  $M$  is  $\sigma_d$ -semistable (resp.  $\sigma_d$ -stable) if and only if  $\sigma_d(d_M) = 0$  and  $\sigma_d(f) \leq 0$  (resp.  $\sigma_d(f) < 0$ ) whenever  $M$  has a proper non-zero subobject of dimension vector  $f$ . We will see that this notion of semistability is related to perpendicular subcategories. Observe first that for any dimension vector  $d$ , and a positive integer  $m$ , we have  $\mathcal{A}(d) \subseteq \mathcal{A}(md)$ . The other inclusion is not true, in general, if the canonical decomposition of  $d$  involves an imaginary Schur root.

**Proposition 3.6.** *Let  $d$  be a dimension vector. The (simple) objects in  $\cup_{m \geq 1} \mathcal{A}(md)$  are the  $\sigma_d$ -(semi)stable objects. If the canonical decomposition of  $d$  does not involve imaginary Schur roots, then  $\cup_{m \geq 1} \mathcal{A}(md) = \mathcal{A}(d)$ .*

*Proof.* Let  $M \in \mathcal{A}(d)$ . Then there exists  $M(d) \in \text{rep}(Q, d)$  such that

$$\text{Hom}(M, M(d)) = 0 = \text{Ext}^1(M, M(d)).$$

Therefore, the semi-invariant  $C^-(M(d))$  of weight  $\sigma_d$  in  $\text{SI}(Q, d_M)$  does not vanish at  $M$ . Since there exists a semi-invariant in  $\text{SI}(Q, d_M)_{\sigma_d}$  that does not vanish on  $M$ ,  $M$  is  $\sigma_d$ -semistable. Conversely, assume that  $M$  is  $\sigma_d$ -semistable. Then there exists  $m \geq 1$  and a semi-invariant  $f \in \text{SI}(Q, d_M)_{m\sigma_d}$  that does not vanish on  $M$ . Now,  $f$  is given by a semi-invariant of the form  $C^-(N)$  for some representation  $N$  of dimension vector  $md$ . We see that  $M$  lies in  $\mathcal{A}(md) \subseteq \cup_{i \geq 1} \mathcal{A}(id)$ . For  $\sigma_d$ -stable representations, one just needs to use king's criterion, as mentioned in the paragraph before this proposition.

Assume now that the canonical decomposition of  $d$  does not involve imaginary Schur root. It follows from Proposition 3.3 that a general representation of dimension vector  $md$  has a direct summand of dimension vector  $d$ . Let  $M \in \mathcal{A}(md)$ .

Then there exists  $N \in \text{rep}(Q, md)$  such that  $C^M(N) \neq 0$ . Also,  $N$  can be taken to be in general position. By the above observation, there is a summand  $N'$  of  $N$  of dimension vector  $d$ . Thus  $C^M(N') \neq 0$ , meaning that  $M \in \mathcal{A}(d)$ . Therefore,  $\mathcal{A}(md) = \mathcal{A}(d)$  for all  $m \geq 1$ .  $\square$

The following proposition explains why the study of perpendicular subcategories is directly related to the study of semi-invariants.

**Proposition 3.7.** *The ring  $\text{SI}(Q, d)$  is generated by the semi-invariants  $C^V(-)$  where  $V$  is simple in  $\mathcal{A}(d)$ .*

*Proof.* We know from Theorem 2.1 that the ring  $\text{SI}(Q, d)$  is generated, over  $k$ , by semi-invariants of the form  $C^V(-)$  where  $V$  is a representation with  $\langle d_V, d \rangle = 0$ . If  $\text{Hom}(V, M) \neq 0$  for all  $M \in \text{rep}(Q, d)$ , then  $C^V(-)$  is the zero semi-invariant. Otherwise,  $V$  lies in  $\mathcal{A}(d)$ . If  $V$  is not a simple object in  $\mathcal{A}(d)$ , then there exists a simple subobject  $V_1$  of  $V$  in  $\mathcal{A}(d)$ . Since  $\mathcal{A}(d)$  is thick, this yields a short exact sequence

$$0 \rightarrow V_1 \rightarrow V \rightarrow V_2 \rightarrow 0$$

in  $\mathcal{A}(d)$  and it follows from [6, Lemma 1] that  $C^V(-) = aC^{V_1}(-)C^{V_2}(-)$  where  $a \in k$  is non-zero. Repeating this reduction for the object  $V_2$  yields that  $C^V(-)$  is, up to a scalar, a product of semi-invariants as in the statement.  $\square$

If  $\sigma$  is a weight, a dimension vector  $d$  is called  $\sigma$ -semistable (resp.  $\sigma$ -stable) if a general representation of dimension vector  $d$  is  $\sigma$ -semistable (resp.  $\sigma$ -stable). Given a  $\sigma$ -semistable dimension vector  $d$ , there exists a  $\sigma$ -stable decomposition  $d = d_1 + \dots + d_r$  of  $d$  where a general representation  $M$  of dimension vector  $d$  has a filtration

$$0 \subset M_1 \subset \dots \subset M_r = M$$

such that for  $1 \leq i \leq r$ ,  $M_i/M_{i-1}$  has dimension vector  $d_i$  and is  $\sigma$ -stable; see [5]. In this case,  $d_i$  will be called a  $\sigma$ -stable factor of  $d$ . The  $\sigma$ -stable decomposition of a  $\sigma$ -semistable dimension vector is unique up to ordering. Moreover, it was shown in [5] that the pairwise distinct  $\sigma$ -stable factors  $f_1, \dots, f_s$  of  $d$  can be ordered in such a way that  $f_i \perp f_j$  whenever  $i < j$ . So the sequence  $(f_1, \dots, f_s)$  is an orthogonal sequence of Schur roots.

**Lemma 3.8.** *Let  $\sigma$  be a weight and  $d$  be a Schur root that is  $\sigma$ -semi-stable. If  $d$  is real, then all the  $\sigma$ -stable factors of  $d$  are real Schur roots. If  $d$  is isotropic, then all the  $\sigma$ -stable factors of  $d$  are real or isotropic Schur roots.*

*Proof.* Let  $d_1, \dots, d_r$  be the distinct  $\sigma$ -stable factors of  $d$ . We may assume that  $(d_1, \dots, d_r)$  is an orthogonal sequence of Schur roots. We can use the algorithm in [4] for finding the canonical decomposition of  $d$  starting with the sequence  $(d_1, \dots, d_r)$ . If one of  $d_i$  is imaginary (or isotropic), it follows from the algorithm that  $d$  will be imaginary (resp. isotropic or imaginary). Therefore, if  $d$  is real, then all  $d_i$  are real. If  $d$  is isotropic, then no  $d_i$  is imaginary.  $\square$

Let  $E = (X_1, \dots, X_r)$  be an exceptional sequence and consider the subcategory  $\mathcal{C}(E) = \mathcal{C}(X_1, \dots, X_r)$ . Assume that the dimension vector  $d$  lies in  $\mathcal{C}(E)$ , that is, there is an object in  $\mathcal{C}(E)$  having  $d$  as dimension vector. The stability condition  $\sigma_d = -\langle -, d \rangle$  also gives rise to a stability condition, denoted  $\sigma_{\mathcal{C}(E), d}$ , in  $\mathcal{C}(E)$ . An object  $X \in \mathcal{C}(E)$  is said to be *relative  $\sigma_d$ -semistable* (resp. *relative  $\sigma_d$ -stable*) in  $\mathcal{C}(E)$  provided it is  $\sigma_{\mathcal{C}(E), d}$ -semistable (resp.  $\sigma_{\mathcal{C}(E), d}$ -stable).

**Proposition 3.9.** *Let  $E = (X_1, \dots, X_r)$  be an exceptional sequence and let  $d$  be a dimension vector lying in  $\mathcal{C}(E)$ . Let  $X \in \mathcal{C}(E)$ .*

- (1) *The object  $X$  is relative  $\sigma_d$ -semistable in  $\mathcal{C}(E)$  if and only if it is  $\sigma_d$ -semistable.*
- (2) *If  $X$  is  $\sigma_d$ -stable, then  $X$  is relative  $\sigma_d$ -stable in  $\mathcal{C}(E)$ .*

*Proof.* Observe that  $X$  is  $\sigma_d$ -semistable if and only if  $-\langle d_X, d \rangle = 0$  and for any subobject  $X'$  of  $X$ , we have  $-\langle d_{X'}, d \rangle \leq 0$ . Similarly,  $X$  is  $\sigma_{\mathcal{C}(E), d}$ -semistable if and only if  $-\langle d_X, d \rangle = 0$  and for any subobject  $X'$  of  $X$  in  $\mathcal{C}(E)$ , we have  $-\langle d_{X'}, d \rangle \leq 0$ . Now, a subobject of  $X$  in  $\mathcal{C}(E)$  is also a subobject of  $X$  in  $\text{rep}(Q)$ . This shows the sufficiency of (1). Suppose now that  $X$  is  $\sigma_{\mathcal{C}(E), d}$ -semistable. Then there exists  $M$  in  $\mathcal{C}(E)$  of dimension vector  $md$  for some  $m > 0$  such that  $C^X(M) \neq 0$ , that is,  $\text{Hom}(X, M) = \text{Ext}^1(X, M) = 0$ . Since  $M$  has dimension vector  $md$  in  $\text{rep}(Q)$ , this gives that  $X$  is  $\sigma$ -semistable. Assume now that  $X$  is  $\sigma$ -stable. Then  $X$  is  $\sigma_{\mathcal{C}, d}$ -semistable. If  $X$  is not  $\sigma_{\mathcal{C}, d}$ -stable, then there exists a proper subobject  $X'$  of  $X$  in  $\mathcal{C}(E)$  such that  $-\langle d_{X'}, d \rangle = 0$ . Now,  $X'$  is also a subobject of  $X$  in  $\text{rep}(Q)$  with  $-\langle d_{X'}, d \rangle = 0$ , and this contradicts the fact that  $X$  is  $\sigma_d$ -stable.  $\square$

#### 4. THE CASE OF AN ISOTROPIC SCHUR ROOT

In this section,  $\delta$  stands for an isotropic Schur root. The weight  $\sigma_\delta$  will simply be denoted  $\sigma$ , when there is no risk of confusion. Our aim is to describe all simple objects in  $\mathcal{A}(\delta)$  or, equivalently, all  $\sigma_\delta$ -stable objects. We start with the following proposition; see [15].

**Proposition 4.1.** *There exists an exceptional sequence  $(V, W)$  in  $\text{rep}(Q)$  such that  $\mathcal{C}(V, W)$  is tame and  $\delta = d_V + d_W$  with  $\langle d_W, d_V \rangle = -2$ . In particular,  ${}^\perp V \cap {}^\perp W \subseteq \mathcal{A}(\delta)$ .*

*Proof.* As shown in [15], there is an exceptional sequence  $E = (V, W)$  of length two such that  $\delta$  is a root in  $\mathcal{C}(E)$ . Since  $E$  has length two,  $\mathcal{C}(E)$  is equivalent to the category of representations of an acyclic quiver  $Q_E$  with two vertices. But  $Q_E$  has to have an isotropic Schur root. Therefore,  $Q_E$  is the Kronecker quiver. With no loss of generality, we may assume that  $V, W$  are the simple objects of  $\mathcal{C}(E)$ . Therefore,  $\langle d_W, d_V \rangle = -\dim_k \text{Ext}^1(V, W)$  and  $\dim_k \text{Ext}^1(V, W)$  is the number of arrows in  $Q_E$ . The second part of the statement is trivial.  $\square$

**Lemma 4.2.** *Let  $Q$  have at least three vertices. Then there is at least one exceptional simple object in  $\mathcal{A}(\delta)$ .*

*Proof.* Suppose that all the  $\sigma$ -stable dimension vectors are isotropic or imaginary. By Proposition 4.1, we have an exceptional sequence  $(V, W)$ , such that  $\delta = d_V + d_W$ . We can extend this to an exceptional sequence  $(U, V, W)$  and hence,  $U \in \mathcal{A}(\delta)$ . This means that  $d_U$  is  $\sigma$ -semistable. Now, we apply Lemma 3.8.  $\square$

The subcategories of the form  $\mathcal{C}(A, B, C)$  where  $(A, B, C)$  is an exceptional sequence will play a crucial role in our investigation. The following lemma, which is easy to check, provides a description of the quivers with three vertices having an isotropic Schur root.

**Lemma 4.3.** *Let  $(A, B, C)$  be an exceptional sequence such that  $\mathcal{C}(A, B, C)$  contains  $\delta$ . Then either*

- (1) *the category  $\mathcal{C}(A, B, C)$  is wild and connected or*

- (2) the category  $\mathcal{C}(A, B, C)$  is equivalent to  $\text{rep}(Q')$ , where  $Q'$  is either of type  $\tilde{\mathbb{A}}_{2,1}$  or a union of the Kronecker quiver and a single vertex.

Let us denote by  $\tau$  the Auslander-Reiten translation in  $\text{rep}(Q)$ . An indecomposable representation that lies in the  $\tau$ -orbit of a projective (resp. injective) representation is called *preprojective* (resp. *preinjective*). Baer and Strauß have proven the following crucial result; see [1] or [21, Theorem B].

**Lemma 4.4** (Baer, Strauß). *Let  $Q$  be of wild type and let  $X$  be exceptional. If  $X^\perp$  is of finite or tame type, then  $X$  has to be preprojective or preinjective.*

The following result describes a way to produce other isotropic Schur roots starting with an exceptional sequence  $(U, V, W)$  where  $\delta = d_V + d_W$ .

**Lemma 4.5.** *Let  $E = (U, V, W)$  be an exceptional sequence such that  $\mathcal{C}(V, W)$  is tame with isotropic Schur root  $\delta = d_V + d_W$ . Reflect both  $V, W$  to the left of  $U$  to get an exceptional sequence  $(V', W', U)$ . Let  $\delta'$  be the unique isotropic Schur root in  $\mathcal{C}(V', W')$ . Then  $\delta' = \delta - \langle \delta, d_U \rangle d_U$ .*

- (1) *If  $U$  is preinjective in  $\mathcal{C}(E)$ , then  $Q$  is wild connected and  $\langle \delta, d_U \rangle \geq 0$ .*
- (2) *If  $U$  is preprojective in  $\mathcal{C}(E)$ , then  $Q$  is wild connected and  $\langle \delta, d_U \rangle \leq 0$ .*
- (3) *Otherwise,  $\mathcal{C}(E)$  is tame,  $U$  is regular or is simple projective-injective in  $\mathcal{C}(E)$ ,  $\langle \delta, d_U \rangle = 0$  and  $\delta' = \delta$ .*

*Proof.* We may assume that  $\text{rep}(Q) = \mathcal{C}(E)$ . If  $Q$  is wild, then it follows from Lemma 4.3 that  $Q$  is connected. In this case, by Lemma 4.4,  $U$  cannot be regular. Assume first that  $Q$  is tame. If  $Q$  is tame connected,  $Q$  is of type  $\tilde{\mathbb{A}}_{2,1}$  and  $U$  has to be isomorphic to one of the two quasi-simple regular exceptional representations. If  $Q$  is tame disconnected, then we see that  $U$  is the simple representation corresponding to the connected component of  $Q$  of type  $\mathbb{A}_1$ . So if  $Q$  is tame, it is clear that  $\langle \delta, d_U \rangle = 0$  and  $\delta = \delta'$ . So we may assume that  $Q$  is wild connected. We have an orthogonal sequence of Schur roots  $(d_U, \delta)$ . Set  $\delta'' = \delta - \langle \delta, d_U \rangle d_U$ . One easily checks that  $\langle \delta'', d_U \rangle = 0$  and  $\langle \delta'', \delta'' \rangle = 0$ . Since  ${}^\perp U = \mathcal{C}(V', W')$  is of tame type, the Euler-Ringel form  $\langle -, - \rangle_{\mathcal{C}(V', W')}$  restricted to  $\mathcal{C}(V', W')$  is positive semi-definite. Therefore,  $\delta''$  is an integral multiple of the isotropic root  $\delta'$  in  $\mathcal{C}(V', W')$ . Since each of  $\delta', \delta''$  is a sum or difference of  $d_{V'}, d_{W'}$ , we see that  $\delta'' = \pm \delta'$ . Assume first that  $U$  is preinjective. A general representation  $M(\delta)$  of dimension vector  $\delta$  is regular while a general representation of dimension vector  $d_U$  is isomorphic to  $U$  hence preinjective. We have  $\text{Ext}^1(M(\delta), U) = 0$  and hence  $\langle \delta, d_U \rangle = \dim_k \text{Hom}(M(\delta), U) \geq 0$ . If  $U$  is preprojective, then  $\text{Hom}(M(\delta), U) = 0$ . This gives  $\langle \delta, d_U \rangle \leq 0$ . Suppose that  $\delta' = \langle \delta, d_U \rangle d_U - \delta$ . This is only possible if  $\langle \delta, d_U \rangle > 0$  and hence, if  $U$  is preinjective. Then  $\langle \delta, d_U \rangle d_U = \delta + \delta'$ . The region  $\mathcal{R}$  given by  $\langle d, d \rangle \leq 0$  is the cone over a two dimensional ellipse and hence, is a convex cone. Since both  $\delta, \delta'$  lie on the boundary of  $\mathcal{R}$ , we see that  $d_U$  lies in  $\mathcal{R}$ . Hence,  $\langle d_U, d_U \rangle \leq 0$ , a contradiction to  $U$  being exceptional. Therefore,  $\delta'' = \delta'$  is the wanted isotropic Schur root.  $\square$

The root  $\delta'$  defined above will be denoted  $L_U(\delta)$ , as it is the reflection of  $\delta$  to the left of the real Schur root  $d_U$ .

**Proposition 4.6.** *Let  $Q$  be wild with three vertices, and let  $(V, W)$  be an exceptional sequence of tame type containing  $\delta$ . Complete this to a full exceptional sequence  $(U, V, W)$ . Then  $U$  is preprojective or preinjective.*

- (1) If  $U$  is preprojective, then the  $\sigma$ -stable representations are, up to isomorphism,  $U$  or  $M(\delta)$  where  $M(\delta)$  is a general representation of dimension vector  $\delta$  in  $\mathcal{C}(V, W)$ . In particular, the  $\sigma$ -stable dimension vectors are  $d_U, \delta$ .
- (2) If  $U$  is preinjective, then the  $\sigma$ -stable representations are, up to isomorphism,  $U$  or  $M(\delta')$  where  $M(\delta')$  is a general representation of dimension vector  $\delta' = L_U(\delta)$  in  $\mathcal{C}(V', W')$ , where  $(V', W', U)$  is exceptional. In particular, the  $\sigma$ -stable dimension vectors are  $d_U, \delta'$ .

In both cases, the  $\sigma$ -stable dimension vectors are the two extremal rays in the cone of  $\sigma$ -semistable dimension vectors.

*Proof.* Recall that the region  $\langle d, d \rangle \leq 0$  is the cone over a two dimensional ellipse. The ray  $[\delta]$  lies on the quadric  $\langle d, d \rangle = 0$ . Since  $\mathcal{C}(V, W)$  is tame, the Euler-Ringel form for  $\mathcal{C}(V, W)$  is positive semi-definite. In particular, any linear combination of  $d_V, d_W$  that lie on the region  $\langle d, d \rangle \leq 0$  has to be a multiple of  $\delta$ . Therefore, the cone  $C_{V,W}$  generated by  $d_V, d_W$  is the tangent plane of  $\langle d, d \rangle = 0$  at  $\delta$ . If  $U'$  is exceptional and  $\sigma$ -semistable, then we have an exceptional sequence  $(U', X, Y)$  such that  $[\delta]$  lies in the cone  $C_{X,Y}$  generated by  $d_X, d_Y$ . Therefore, as argued above,  $C_{X,Y}$  is the tangent plane of  $\langle d, d \rangle = 0$  at  $\delta$ . Hence,  $C_{V,W} = C_{X,Y}$  and thus, since  $\langle -, - \rangle$  is non-degenerate, the rays of  $d_U, d_{U'}$  coincide. This gives  $U \cong U'$  since  $U, U'$  are exceptional. This proves that there is a unique  $\sigma$ -semistable real Schur root  $d_U$  and a unique  $\sigma$ -semistable exceptional representation, up to isomorphism. Hence,  $d_U$  has to be  $\sigma$ -stable by Lemma 4.2. Observe that  $Q$  is connected by Lemma 4.3. By Lemma 4.4, since  $\mathcal{C}(V, W)$  is tame, we know that  $U$  is preinjective or preprojective. Let  $M$  be a  $\sigma$ -stable representation that is not isomorphic to  $U$ . By the previous argument,  $M$  cannot be exceptional. Suppose first that  $U$  is preprojective. If  $M$  is not in  $U^\perp$ , then  $\text{Ext}^1(U, M) \neq 0$  as  $\text{Hom}(U, M) = 0$ . Therefore,  $M$  has to be preprojective, contradicting that  $M$  is not exceptional. Hence,  $M$  is  $\sigma$ -stable and lies in  $U^\perp = \mathcal{C}(V, W)$ . By Proposition 3.9,  $M$  is relative  $\sigma$ -stable in  $U^\perp$ . Therefore,  $M \cong M(\delta)$  is a general representation of dimension vector  $\delta$  in  $\mathcal{C}(V, W)$ . Suppose finally that  $U$  is preinjective. If  $M$  is not in  ${}^\perp U$ , then  $\text{Ext}^1(M, U) \neq 0$  and this means that  $M$  is preinjective, thus exceptional, a contradiction. Therefore,  $M$  lies in  ${}^\perp U$ . Now,  ${}^\perp U$  is also tame. Consider the exceptional sequence  $(V', W', U)$  where  $V', W'$  are obtained from  $V, W$  by reflecting to the left of  $U$ . Let  $\delta' = L_U(\delta)$  be the unique isotropic Schur root in  $\mathcal{C}(V', W')$ . Since  $M$  is Schur and not exceptional,  $M$  has dimension vector  $\delta'$ .  $\square$

Let  $E$  be an exceptional sequence of  $\sigma$ -stable representations. Clearly, such a sequence has length at most  $n - 2$ . The sequence  $E$  is said to be *full* if  $E$  has length  $n - 2$ . The following result guarantees that such a full exceptional sequence of  $\sigma$ -stable representations always exists.

**Lemma 4.7.** *Any exceptional sequence of  $\sigma$ -stable representations can be completed to a full exceptional sequence of  $\sigma$ -stable representations. In particular, there exists an exceptional sequence  $(M_{n-2}, \dots, M_2, M_1)$  of  $\sigma$ -stable representations.*

*Proof.* Let  $(X_1, \dots, X_r)$  be an exceptional sequence with all  $X_i$   $\sigma$ -stable. Assume that  $r$  is not equal to  $n - 2$ . Observe that  $\langle d_{X_i}, d_{X_j} \rangle \leq 0$  whenever  $i \neq j$ . By Proposition 4.1, and since the perpendicular category  $\mathcal{C}(X_1, \dots, X_r)^\perp$  contains representations of dimension vector  $\delta$ , we can extend it to an exceptional sequence

$$(X_1, \dots, X_r, V, W)$$

of length less than  $n$  where  $\mathcal{C}(V, W)$  is tame and  $\delta = d_V + d_W$ . Therefore, there exists an exceptional representation  $Y$  such that  $(X_1, \dots, X_r, Y, V, W)$  is exceptional. Being left orthogonal to both  $V, W$ , the representation  $Y$  has to be  $\sigma$ -semistable. It is well known and easy to see that for any acyclic quiver, there exists a sincere representation of it that is rigid. Since the category  $\mathcal{C}(X_1, \dots, X_r, Y)$  is equivalent to the category of representations of an acyclic quiver with  $r + 1$  vertices, there exists a rigid object  $Z \in \mathcal{C}(X_1, \dots, X_r, Y)$  such that its Jordan-Hölder composition factors in  $\mathcal{C}(X_1, \dots, X_r, Y)$  will consist of all the simple objects in  $\mathcal{C}(X_1, \dots, X_r, Y)$ . Being  $\sigma$ -stable, the objects  $X_1, \dots, X_r$  are non-isomorphic simple objects in  $\mathcal{C}(X_1, \dots, X_r, Y)$ . Let  $Y'$  be the other simple object. Being rigid,  $Z$  is a general representation. It has some filtration where the subquotients are  $\{X_1, \dots, X_r, Y'\}$  with possible multiplicities. If  $Y'$  is  $\sigma$ -stable, then we have a list  $\{X_1, \dots, X_r, Y'\}$  of  $r + 1$  objects that are  $\sigma$ -stable. Since these objects are the simple objects of  $\mathcal{C}(X_1, \dots, X_r, Y)$ , they could be ordered to form an exceptional sequence of length  $r + 1$ , which will be the wanted sequence. If  $Y'$  is not  $\sigma$ -stable, then the above filtration of  $Z$  can be refined to a filtration in  $\text{rep}(Q)$  where all subquotients are  $\sigma$ -stable. In particular, there will be more than  $r$  non-isomorphic subquotients. Since  $Z$  is rigid, we know that all subquotients have dimension vector a real Schur root, by Lemma 3.8. Now, these real Schur roots can be ordered to form an orthogonal sequence of real Schur roots; see [5]. This sequence corresponds to an exceptional sequence of  $\sigma$ -stable representations of length at least  $r + 1$ .  $\square$

Now, we treat the case where  $n$  is arbitrary. We first need some more notations. From now on, let us fix  $(M_{n-2}, \dots, M_1)$  a full exceptional sequence of  $\sigma$ -stable representations. Complete this sequence to get a full exceptional sequence  $(M_{n-2}, \dots, M_1, V, W)$ . Note that  $\delta$  is a root in  $\mathcal{C}(V, W)$ . We construct a sequence of isotropic Schur roots  $\delta_1, \dots, \delta_{n-2}$  and a sequence of rank three subcategories  $\mathcal{C}_i$  as follows. Set  $\delta_1 = \delta$ ,  $V_1 = V$  and  $W_1 = W$  and  $\mathcal{C}_1 = \mathcal{C}(M_1, V_1, W_1)$ . Observe that if  $\mathcal{C}_1$  is wild, then it is connected and  $M_1$  is preprojective or preinjective in  $\mathcal{C}_1$ . If  $\mathcal{C}_1$  is tame, then it is either disconnected (in which case  $M_1$  is the unique simple object in the trivial component of  $\mathcal{C}_1$ ) or else it is connected and  $M_1$  is quasi-simple in  $\mathcal{C}_1$ . For  $1 \leq i \leq n - 2$ , if  $M_i$  is not preinjective in  $\mathcal{C}_i = \mathcal{C}(M_i, V_i, W_i)$ , then we reduce to the case of the exceptional sequence  $(M_{n-2}, \dots, M_{i+1}, V_{i+1}, W_{i+1})$  and we set  $\delta_{i+1} = \delta_i$ ,  $V_{i+1} = V_i$ ,  $W_{i+1} = W_i$ . If  $M_i$  is preinjective in  $\mathcal{C}(M_i, V_i, W_i)$ , we reduce to the case of the exceptional sequence  $(M_{n-2}, \dots, M_{i+1}, V_{i+1}, W_{i+1})$  where  $V_{i+1}$  is the reflection of  $V_i$  to the left of  $M_i$  and  $W_{i+1}$  is the reflection of  $W_i$  to the left of  $M_i$ . We set  $\delta_{i+1} = \delta_i - \langle \delta_i, d_{M_i} \rangle d_{M_i}$ . In all cases, we set  $\mathcal{C}_{i+1} = \mathcal{C}(M_{i+1}, V_{i+1}, W_{i+1})$ . Note that for  $1 \leq i \leq n - 2$ , the root  $\delta_i$  is a root in  $\mathcal{C}(V_i, W_i)$ . Finally, we set  $\bar{\delta} = \delta_{n-2}$ .

For two dimension vectors  $d_1, d_2$ , we write  $d_1 \hookrightarrow d_2$  provided a general representation of dimension vector  $d_2$  has a subrepresentation of dimension vector  $d_1$ .

**Lemma 4.8.** *We have  $\delta_{i+1} \hookrightarrow \delta_i$  and each  $\delta_i$  is an isotropic Schur root that is  $\sigma$ -semistable.*

*Proof.* We proceed by induction on  $i$ . The case where  $i = 1$  is clear. Let  $M(\delta_i)$  be a general representation of dimension vector  $\delta_i$  in  $\mathcal{C}(V_i, W_i)$  that is  $\sigma$ -semistable. If  $M_i$  is not preinjective in  $\mathcal{C}_i = \mathcal{C}(M_i, V_i, W_i)$ , then  $\delta_{i+1} = \delta_i$  and hence, it is clear that  $\delta_{i+1} \hookrightarrow \delta_i$  and that  $\delta_i$  is an isotropic Schur root that is  $\sigma$ -semistable. So

assume that  $M_i$  is preinjective in  $\mathcal{C}_i$ . Then  $\text{Ext}^1(M(\delta_i), M_i) = 0$ . If we have a non-zero morphism  $f_i : M(\delta_i) \rightarrow M_i$  that is not an epimorphism, then the cokernel  $C_i$  of  $f_i$  is  $\mathcal{C}_i$ -preinjective and  $\sigma$ -semistable in  $\mathcal{C}_i$ . There is an epimorphism  $C_i \rightarrow N_i$  where  $N_i$  is relative  $\sigma$ -stable in  $\mathcal{C}_i$ . Since  $M_i$  is  $\mathcal{C}_i$ -preinjective,  $N_i$  (and  $C_i$ ) has to be  $\mathcal{C}_i$ -preinjective and the only  $\mathcal{C}_i$ -preinjective relative  $\sigma$ -stable object in  $\mathcal{C}_i$  is  $M_i$ . This is a contradiction. Therefore, any non-zero morphism  $M(\delta_i) \rightarrow M_i$  is an epimorphism. Now, take  $e_i = \langle \delta_i, d_{M_i} \rangle = \dim \text{Hom}(M(\delta_i), M_i)$ . We have a morphism  $g_i : M(\delta_i) \rightarrow M_i^{e_i}$  given by a basis of  $\text{Hom}(M(\delta_i), M_i)$ , and as argued above,  $g_i$  is an epimorphism. The kernel  $K_i$  of  $g_i$  is of dimension  $\delta_i - \langle \delta_i, d_{M_i} \rangle d_{M_i}$  which is  $\delta_{i+1}$ . Hence,  $\delta_{i+1}$  is relative  $\sigma$ -semistable and hence  $\sigma$ -semistable. Consider the short exact sequence

$$0 \rightarrow K_i \rightarrow M(\delta_i) \rightarrow M_i^{e_i} \rightarrow 0$$

in  $\mathcal{C}_i$ . Since  $K_i \in {}^\perp M_i$  by construction, and  $K_i$  has dimension vector  $\delta_{i+1}$ , we have  $\text{ext}(\delta_{i+1}, e_i \cdot d_{M_i}) = 0$  and by Schofield's result [17, Theorem 3.3],  $\delta_{i+1} \hookrightarrow \delta_{i+1} + e_i \cdot d_{M_i} = \delta_i$ .  $\square$

Call  $\delta$  of *smaller type* if the  $\tau$ -orbit of a general representation of dimension vector  $\delta$  contains a non-sincere representation. Equivalently, if  $\tau$  denotes the Coxeter matrix of  $Q$ , then there is an integer  $r$  such that  $\tau^r \delta$  is not sincere.

**Proposition 4.9.** *The following conditions are equivalent.*

- (1) *The root  $\delta$  is of smaller type.*
- (2) *There is a  $\sigma$ -stable representation that is preprojective or preinjective.*
- (3) *There is a  $\sigma$ -semistable representation that is preprojective or preinjective.*

*Proof.* Let  $Z$  be a  $\sigma$ -semistable representation that is preprojective or preinjective. If  $Z$  is preprojective, then there is some  $i \geq 0$  with  $\tau^i Z = P$  projective. Since  $\langle d_Z, \delta \rangle = 0$ , we get  $\langle d_P, \tau^i \delta \rangle = 0$ , showing that  $\tau^i \delta$  is not sincere, that is,  $\delta$  is of smaller type. If  $Z$  is preinjective, then there is some  $j < 0$  with  $\tau^j Z = Q[1]$  a shift of a projective representation  $Q$ . Since  $\langle d_Z, \delta \rangle = 0$ , we get  $\langle -d_Q, \tau^j \delta \rangle = 0$ , showing that  $\tau^j \delta$  is not sincere, that is,  $\delta$  is of smaller type. This proves that (3) implies (1). Clearly, (2) implies (3). Assume that  $\delta$  is of smaller type. Then there is some integer  $i$  and some  $x \in Q_0$  such that  $\text{Hom}(P_x, \tau^i M(\delta)) = 0$  whenever  $M(\delta)$  has dimension vector  $\delta$ . Therefore, there exists a preprojective (if  $i \geq 0$ ) or preinjective (if  $i < 0$ ) representation  $Z$  such that  $Z$  is left orthogonal to any representation  $M(\delta)$  of dimension vector  $\delta$ . Therefore  $Z$  is  $\sigma$ -semistable. If  $Z$  is not  $\sigma$ -stable and preprojective, then it has a  $\sigma$ -stable subrepresentation  $Z'$  which has to be preprojective. If  $Z$  is not  $\sigma$ -stable and preinjective, then it has a  $\sigma$ -stable quotient  $Z'$  which has to be preinjective.  $\square$

We start with the following result, which describes the  $\sigma$ -stable objects when  $n = 4$ . The core of it will be generalized to arbitrary  $n$  later. We think it is interesting to have a separate result for the  $n = 4$  case since more can be said for this small case.

**Proposition 4.10.** *Let  $Q$  be a connected wild quiver with 4 vertices and let  $(M_2, M_1)$  be a full exceptional sequence of  $\sigma$ -stable representations. Consider the exceptional sequence  $(M_2, M_1, V, W)$  with  $V, W$  the simple objects in  $\mathcal{C}(V, W)$ . Let  $M$  be  $\sigma$ -stable but not isomorphic to  $M_1$  or  $M_2$ . Then*

- (1) *The root  $\bar{\delta}$  is the only  $\sigma$ -stable non-real Schur root.*

- (2) *The root  $\delta$  is of smaller type if and only if one of  $M_1, M_2$  is preprojective or preinjective.*  
 (3) *If  $\delta$  is not of smaller type, then the only exceptional  $\sigma$ -stable representations are  $M_1, M_2$ .*

*Proof.* The sufficiency of (2) follows from Proposition 4.9. Assume now that both  $M_1, M_2$  are regular. We claim that  $\delta$  is not of smaller type and that the only exceptional  $\sigma$ -stable representations are  $M_1, M_2$ . In this case, by Lemma 4.4, since  $\mathcal{C}_1 = \mathcal{C}(M_1, V, W)$  is wild and  $\mathcal{C}(V, W)$  is tame, we know that  $M_1$  is either preprojective or preinjective in  $\mathcal{C}_1$ . Observe that  $\mathcal{C}_1$  is connected since it is wild and  $\delta$  is an isotropic root in  $\mathcal{C}_1$ . If  $M_1$  is preinjective in  $\mathcal{C}_1$ , then  $\mathcal{C}_2 = {}^\perp M_1$  is also wild, since  $M_1$  is regular. If  $M_1$  is preprojective in  $\mathcal{C}_1$ , consider the exceptional sequence  $(M'_1, M_2, V, W) = (M'_1, M_2, V_2, W_2)$ . Note that  $M_1, M_2$  are the relative simples in  $\mathcal{C}(M_2, M_1)$ . Since any indecomposable object in  $\mathcal{C}(M_2, M_1)$  has a morphism from and to  $M_1 \oplus M_2$ , all objects of  $\mathcal{C}(M_2, M_1)$ , seen as objects in  $\text{rep}(Q)$ , are regular. Therefore,  $M'_1 \in \mathcal{C}(M_2, M_1)$  is regular in  $\text{rep}(Q)$  and  $M'_1{}^\perp = \mathcal{C}(M_2, V_2, W_2) = \mathcal{C}_2$  is wild (and connected). Let  $M$  be an arbitrary  $\sigma$ -stable representation not isomorphic to  $M_1$  or  $M_2$ . Since  $\text{Hom}(M_2, M) = 0$ , there exists a non-negative integer  $d_2$  and a short exact sequence

$$(*) : \quad 0 \rightarrow M \rightarrow E_2 \xrightarrow{f_1} M_2^{d_2} \rightarrow 0$$

with  $E_2 \in M_2^\perp$ . Observe that  $d_2 = 0$  if and only if  $M \in M_2^\perp$ . Observe also that  $\text{Hom}(E_2, M_1) = \text{Hom}(M_1, E_2) = 0$ . Since  $\text{Hom}(M_1, E_2) = 0$ , there exists a non-negative integer  $d_1$  and a short exact sequence

$$(*) : \quad 0 \rightarrow E_2 \rightarrow E_1 \xrightarrow{f_1} M_1^{d_1} \rightarrow 0$$

with  $E_1 \in M_2^\perp \cap M_1^\perp$ . Observe that  $d_1 = 0$  if and only if  $E_2 \in M_1^\perp$ . Since  $E_1$  is relative  $\sigma$ -semistable in  $M_2^\perp \cap M_1^\perp = \mathcal{C}(V, W)$ , there is a monomorphism  $M(\delta) \rightarrow E_1$  where  $M(\delta)$  is a Schur representation of dimension vector  $\delta$  in  $\mathcal{C}(V, W)$ . If  $M_1$  is preinjective in  $\mathcal{C}_1$ , then  $\text{Ext}^1(M(\delta), M_1) = 0$ . Then  $e_1 := \langle \delta, d_{M_1} \rangle = \dim \text{Hom}(M(\delta), M_1) > 0$ . We have a short exact sequence

$$0 \rightarrow M(\delta_1) \rightarrow M(\delta) \rightarrow M_1^{e_1} \rightarrow 0$$

and this yields a monomorphism  $M(\delta_1) \rightarrow E_2$ . If  $M_1$  is preprojective in  $\mathcal{C}_1$ , then  $\text{Hom}(M(\delta), M_1) = 0$  and hence, we get a monomorphism  $M(\delta_1) = M(\delta) \rightarrow E_2$ . If  $M_2$  is preinjective in  $\mathcal{C}_2$ , then  $\text{Ext}^1(M(\delta_1), M_2) = 0$ . Then  $e_2 := \langle \delta_1, d_{M_2} \rangle = \dim \text{Hom}(M(\delta_1), M_2) > 0$ . We have a short exact sequence

$$0 \rightarrow M(\delta_2) \rightarrow M(\delta_1) \rightarrow M_2^{e_2} \rightarrow 0$$

and this yields a monomorphism  $M(\delta_2) \rightarrow M$ . If  $M_2$  is preprojective in  $\mathcal{C}_2$ , then  $\text{Hom}(M(\delta_1), M_2) = 0$  and hence, we get a monomorphism  $M(\delta_2) = M(\delta_1) \rightarrow M$ . This shows that  $M \cong M(\delta_2) = M(\delta)$ . In particular, the exceptional sequence  $(M_2, M_1)$  of  $\sigma$ -stable representations is unique. It follows from Proposition 4.9 that  $\delta$  is not of smaller type. This prove our claim and hence (2) and (3). Statement (1) in the case where  $\delta$  is of smaller type is a consequence of the next proposition, Proposition 4.11.  $\square$

Given a subcategory  $\mathcal{C}$  of  $\text{rep}(Q)$ , we denote by  $\tau_{\mathcal{C}}$  the Auslander-Reiten translation in  $\mathcal{C}$ . Fix a full exceptional sequence  $(M_{n-2}, \dots, M_2, M_1)$  of  $\sigma$ -stable representations, and complete it to form an exceptional sequence  $(M_{n-2}, \dots, M_2, M_1, V, W)$



where  $V, W$  are the non-isomorphic simple objects in  $\mathcal{C}(V, W)$ . In particular,  $\delta = d_V + d_W$  and  $\langle d_W, d_V \rangle = -2$ . Recall that for  $1 \leq i \leq n-2$ , we denote by  $\mathcal{C}_i$  the subcategory  $\mathcal{C}(M_i, V_i, W_i)$ , where  $V_i, W_i$  have been defined previously. If  $\mathcal{C}_i$  is tame and connected, then  $M_i$  is regular quasi-simple in  $\mathcal{C}_i$  and lies in a tube of rank 2 in  $\mathcal{C}_i$ . We set  $N_i := \tau_{\mathcal{C}_i} M_i = \tau_{\mathcal{C}_i}^{-1} M_i$ . The exceptional sequence  $(M_{n-2}, \dots, M_{i+1})$  is denoted by  $\mathcal{E}_i$ .

**Proposition 4.11.** *Let  $Q$  be a connected quiver with isotropic Schur root  $\delta$ . Let  $N$  be a  $\sigma$ -stable object not isomorphic to any  $M_i$ .*

- (1) *The root  $\bar{\delta}$  is the only  $\sigma$ -stable Schur root that is not real.*
- (2) *If  $N$  is not exceptional, then  $d_N = \bar{\delta}$  and  $N \in \mathcal{C}(V_{n-2}, W_{n-2})$ .*
- (3) *If  $N$  is exceptional, then there exists some  $i$  such that  $\mathcal{C}_i$  is tame connected with  $M_i$  quasi-simple. There exists a subsequence  $\mathcal{F}_i$  of  $\mathcal{E}_i$  such that  $N$  is the reflection of  $N_i$  to the left of  $\mathcal{F}_i$ .*

*Proof.* We first need to check that there is at least one  $\sigma$ -stable root that is not real. Assume otherwise. As  $\delta$  is  $\sigma$ -stable, its  $\sigma$ -stable decomposition will involve only real Schur roots. These roots may be ordered to form a Schur sequence. Since all roots are real, this will correspond to an exceptional sequence  $(F_1, \dots, F_r)$ . Suppose that  $r < n$ . Complete the latter sequence to get a full exceptional sequence  $(F_1, \dots, F_n)$ . Since  $\delta$  is a root in each  $F_i^\perp$  for  $1 \leq i \leq r$ , we see that  $\delta$  is a root in  $\mathcal{C}(F_{r+1}, \dots, F_n)$ . Since  $\delta$  is also a root in  $\mathcal{C}(F_1, \dots, F_r)$ , we get that  $\delta$  is in the span of  $d_{F_1}, \dots, d_{F_r}$  as well as in the span of  $d_{F_{r+1}}, \dots, d_{F_n}$ . Since  $(F_1, \dots, F_n)$  is an exceptional sequence, the vectors  $d_{F_1}, \dots, d_{F_n}$  are linearly independent, a contradiction. Hence,  $r = n$ . But then,  $\mathcal{C}(F_1, \dots, F_n) = \text{rep}(Q)$  and all objects of  $\text{rep}(Q)$  are  $\sigma$ -semistable, which is also a contradiction. In the rest of the proof, we will see that the only possibility for a non-real  $\sigma$ -stable Schur root is  $\bar{\delta} = \delta_{n-2}$ .

Let  $N$  be a  $\sigma$ -stable representation not isomorphic to any  $M_i$ . In particular,  $\text{Hom}(N, M_i) = \text{Hom}(M_i, N) = 0$  for all  $i$ . Let  $E_{n-2} := N$ . For  $i = n-2, n-3, \dots, 2, 1$ , we have a short exact sequence

$$0 \rightarrow E_i \rightarrow E_{i-1} \rightarrow M_i^{d_i} \rightarrow 0$$

where  $d_i$  is a non-negative integer with  $d_i = -\dim \text{Ext}^1(M_i, E_i) = \langle d_{M_i}, d_{E_i} \rangle \geq 0$ , as  $\text{Hom}(M_i, E_i) = 0$  by induction. Observe that for  $i < n-2$ , we have  $E_i \in M_{n-2}^\perp \cap \dots \cap M_{i+1}^\perp$ . In particular,  $E_0$  lies in  $\mathcal{C}(V, W)$ . Observe also that all  $E_i$  are  $\sigma$ -semistable.

Since  $E_0$  is relative  $\sigma$ -semistable in  $\mathcal{C}(V, W)$ , there is a monomorphism  $Z_0 := M(\delta) \rightarrow E_0$  where  $M(\delta)$  is a Schur representation of dimension vector  $\delta$  in  $\mathcal{C}(V, W)$ . If  $M_1$  is preinjective in  $\mathcal{C}_1$ , then  $\text{Ext}^1(M(\delta), M_1) = 0$ . Then  $e_1 := \langle \delta, d_{M_1} \rangle = \dim \text{Hom}(M(\delta), M_1)$ . We have a short exact sequence

$$0 \rightarrow M(\delta_2) \rightarrow M(\delta) \rightarrow M_1^{e_1} \rightarrow 0$$

and this yields a monomorphism  $M(\delta_2) \rightarrow E_1$  where  $M(\delta_2)$  lies in  $\mathcal{C}(V_2, W_2)$  is  $\sigma$ -semistable. If  $M_1$  is preprojective in  $\mathcal{C}_1$ , then  $\text{Hom}(M(\delta), M_1) = 0$  and hence, we get a monomorphism  $M(\delta_2) = M(\delta) \rightarrow E_1$ . If  $\mathcal{C}_1$  is tame disconnected, then  $\text{Hom}(M(\delta), M_1) = 0$  and we get a monomorphism  $M(\delta_2) = M(\delta) \rightarrow E_1$  as well. Assume that  $\mathcal{C}_1$  is tame connected. If  $\text{Hom}(M(\delta), M_1) = 0$ , then we get a monomorphism  $M(\delta_2) = M(\delta) \rightarrow E_1$  as previously. Assume that  $\text{Hom}(M(\delta), M_1) \neq 0$ . Then  $\text{Hom}(N_1, M_1) = 0$  and there is a monomorphism  $N_1 \rightarrow M(\delta)$ . Therefore, we get a monomorphism  $N_1 \rightarrow E_1$ . Hence, in all cases, we have a monomorphism

$Z_1 \rightarrow E_1$  where  $Z_1$  is either a Schur ( $\sigma$ -semistable) representation of dimension vector  $\delta_2$  in  $\mathcal{C}(V_2, W_2)$  or is  $N_1$ . In the first case, we consider the exceptional sequence  $(M_{n-2}, \dots, M_2, V_2, W_2)$  and we proceed by induction starting with the monomorphism  $Z_1 \rightarrow E_1$  where  $Z_1 = M(\delta_2)$ . For the second case, we consider the exceptional sequence  $(M_{n-2}, \dots, M_2, N_1)$ . If  $\text{Hom}(N_1, M_2) \neq 0$ , then  $\text{Ext}^1(N_1, M_2) = 0$  as  $\text{Ext}^1(M_2, N_1) = 0$ . We have a short exact sequence

$$0 \rightarrow Z_2 \rightarrow N_1 \rightarrow M_2^{f_2} \rightarrow 0$$

where  $f_2 = \dim \text{Hom}(N_1, M_2)$ . Then,  $Z_2 \in {}^\perp M_2$  and hence, we get a monomorphism  $Z_2 \rightarrow E_2$  and  $Z_2$  is the reflection of  $N_1$  to the left of  $M_2$ . If  $\text{Hom}(N_1, M_2) = 0$ , then we get a monomorphism  $Z_2 := N_1 \rightarrow E_2$ . We proceed by induction until we get a monomorphism  $Z_{n-2} \rightarrow E_{n-2} = N$ . This gives  $Z_{n-2} \cong N$ . We see that  $Z_{n-2}$  will be of the required form. In particular, if  $N$  is not exceptional, then  $N$  has dimension vector  $\delta_{n-2} = \bar{\delta}$  and will be in  $\mathcal{C}(V_{n-2}, W_{n-2})$ .  $\square$

The next two lemmas will help us in giving a better description of the simple objects in  $\mathcal{A}(\delta)$ , that is, the  $\sigma$ -stable objects.

**Lemma 4.12.** *Assume that both  $\mathcal{C}(M_{t-1}, \dots, M_1, V, W)$  and  $\mathcal{C}_t$  are tame connected with  $\delta$  the unique isotropic Schur root in both of these subcategories. Then  $\mathcal{C}(M_t, M_{t-1}, \dots, M_1, V, W)$  is tame connected with only one isotropic Schur root  $\delta$ .*

*Proof.* Assume that  $\mathcal{C} := \mathcal{C}(M_t, M_{t-1}, \dots, M_1, V, W)$  is wild. Then  $M_t$  lies in the preprojective or preinjective component of a wild connected component of  $\mathcal{C}$ . Observe that, in  $\mathcal{C}_t$ , the object  $M_t$  is left orthogonal to the unique isotropic Schur root (which is  $d_{V_t} + d_{W_t}$ ). Since  $\mathcal{C}_t$  is connected, this means that  $M_t$  is regular in  $\mathcal{C}_t$ . In particular, there are infinitely many indecomposable objects  $Z$  of  $\mathcal{C}_t$  (and hence of  $\mathcal{C}$ ) with  $\text{Hom}(M_t, Z) \neq 0$ . Therefore,  $M_t$  cannot lie in a preinjective component of  $\mathcal{C}$ . Similarly,  $M_t$  cannot lie in a preprojective component of  $\mathcal{C}$ . This is a contradiction. It remains to show that  $\mathcal{C}$  is connected (it is clear that then,  $\delta$  will be the unique isotropic Schur root in  $\mathcal{C}$ ). Assume otherwise. Then, we have  $\mathcal{C} \cong \mathcal{B}_1 \times \mathcal{B}_2$  where each of  $\mathcal{B}_1, \mathcal{B}_2$  is equivalent to a category of representations of a non-empty acyclic quiver. Assume that  $\delta \in \mathcal{B}_2$ . By assumption, we must have that both  $\mathcal{C}(M_{t-1}, \dots, M_1, V, W), \mathcal{C}_t$  are subcategories of  $\mathcal{B}_2$ . But then,  $\mathcal{C} = \mathcal{B}_2$ , a contradiction. Therefore,  $\mathcal{C}$  is connected.  $\square$

**Lemma 4.13.** *Assume that  $\mathcal{C}_i$  is tame connected for all  $1 \leq i \leq s-1$ . Fix  $1 \leq t \leq s-1$  and consider the representation  $N_t$  as defined above.*

- (1) *If  $M_s$  is preprojective in the category  $\mathcal{C}_s$ , then  $\text{Hom}(N_t, M_s) = 0$ .*
- (2) *If  $M_s$  is preinjective in the category  $\mathcal{C}_s$ , then  $M_i \in {}^\perp M_s$  for all  $1 \leq i \leq s-1$ .*
- (3) *If  $M_s$  is simple disconnected in  $\mathcal{C}_s$ , then all  $\sigma$ -stable representations, except possibly  $M_{s-1}, \dots, M_1$ , lie in  ${}^\perp M_s$ .*

*Proof.* Consider the categories

$$\mathcal{C} := \mathcal{C}(M_s, \dots, M_1, V, W)$$

and

$$\mathcal{C}' := \mathcal{C}(M_{s-1}, \dots, M_1, V, W).$$

We know that  $\mathcal{C}'$  is tame connected by Lemma 4.12. Assume that  $\mathcal{C}$  is wild. Therefore,  $M_s$  lies in a preprojective or preinjective component of  $\mathcal{C}$ . For proving (1),

assume that  $M_s$  is preprojective in the category  $\mathcal{C}_s$ , which means that  $M_s$  is preprojective in  $\mathcal{C}$ . Assume to the contrary that  $\text{Hom}(N_t, M_s) \neq 0$ . Since  $N_t \in \mathcal{C}$ , the object  $N_t$  is preprojective in  $\mathcal{C}$ . Therefore,  $N_t$  cannot be regular in  $\mathcal{C}_t$ , a contradiction. For proving (2), assume that  $M_s$  is preinjective in the category  $\mathcal{C}_s$ , which means that  $M_s$  is preinjective in  $\mathcal{C}$ . Since  $t$  is arbitrary and  $\text{Hom}(M_t, M_i) = 0$ , it is sufficient to prove that  $\text{Ext}^1(M_t, M_s) = 0$ . If not, then the Auslander-Reiten formula in  $\mathcal{C}$  yields a nonzero morphism from  $M_s$  to  $\tau_{\mathcal{C}} M_t$  and hence,  $\tau_{\mathcal{C}} M_t$  (and thus  $M_t$ ) is preinjective in  $\mathcal{C}$ . Thus,  $M_t$  cannot be regular in  $\mathcal{C}_t$ , a contradiction. Clearly, if  $\mathcal{C}$  is tame, then (1), (2) cannot occur, since a subcategory of a tame category cannot be wild.

It remains to prove (3). Let  $N$  be a  $\sigma$ -stable object not isomorphic to any  $M_i$ . It follows from the proof of the last theorem that we have a short exact sequence

$$0 \rightarrow E_s \rightarrow E_{s-1} \rightarrow M_s^{d_s} \rightarrow 0$$

and a monomorphism  $Z \rightarrow E_{s-1}$  where  $Z$  is either indecomposable of dimension vector  $\delta$  in  $\mathcal{C}(V, W)$  or is exceptional and relative  $\sigma$ -semistable in  $\mathcal{C}'$  (a reflection of one of the  $N_i$  for  $1 \leq i \leq s-1$ ). In the first case, since  $\mathcal{C}_s$  is tame disconnected, this yields a monomorphism  $Z \rightarrow E_s$  and the rest of the proof of the above theorem deals with representations in  ${}^\perp M_s$ . In the second case, we get a monomorphism  $M \rightarrow Z$  where  $M$  is relative  $\sigma$ -stable in  $\mathcal{C}'$  and exceptional, and hence a quasi-simple object of  $\mathcal{C}'$ . Assume that  $M$  lies in a tube  $T$  of rank  $r$ . In particular, the other non-isomorphic  $r-1$  quasi-simple objects of that tube are among the objects  $M_{s-1}, \dots, M_1$ . We claim that  $\text{Ext}^1(M, M_s) = 0$ . There exists a short exact sequence

$$0 \rightarrow M \rightarrow Y \rightarrow N \rightarrow 0$$

in  $\mathcal{C}'$  where  $M, Y, N$  are indecomposable in  $T$  with  $Y$  of dimension vector  $\delta$  and  $N$  of quasi-length  $r-1$  which does not have  $M$  as a quasi-simple composition factor. Using that  $(M_{s-1}, \dots, M_1, V, W)$  is an exceptional sequence and since, by construction,  $M$  is the unique quasi-simple of  $T$  with  $\text{Hom}(M, Y) \neq 0$ , we have  $Y \in \mathcal{C}(V, W)$ . The surjective map  $\text{Ext}^1(Y, M_s) \rightarrow \text{Ext}^1(M, M_s)$  together with the fact that  $\mathcal{C}_s = \mathcal{C}(M_s, V, W)$  is disconnected gives that  $\text{Ext}^1(M, M_s) = 0$ . This proves our claim. Observe that  $M$  lies in  $M_s^\perp$ . We have a short exact sequence

$$0 \rightarrow Z' \rightarrow M \rightarrow M_s^f \rightarrow 0$$

where  $f = \dim \text{Hom}(M, M_s) = \langle d_M, d_{M_s} \rangle$ . Thus,  $Z'$  is the reflection of  $M$  to the left of  $M_s$  and  $Z' \in {}^\perp M_s$  and we get a monomorphism  $Z' \rightarrow E_s$ . The proof of the last theorem continues with  $Z_s = Z'$  and the monomorphism  $Z_s \rightarrow E_s$ . In all cases, we see that  $M_s$  satisfies the required property.  $\square$

## 5. CONE OF $\sigma$ -SEMI-STABLE DIMENSION VECTORS

Let  $d$  be a dimension vector. Let us denote by  $C(\sigma_d)$  the set of all  $\sigma_d$ -semistable dimension vectors. We consider  $C_{\mathbb{R}}(\sigma_d)$  the corresponding cone in  $\mathbb{R}^n$ , which lie in the positive orthant of  $\mathbb{R}^n$ . Since the rays  $x \in C_{\mathbb{R}}(\sigma_d)$  satisfy  $\langle x, d \rangle = 0$ , we rather consider  $C_{\mathbb{R}}(\sigma_d)$  as a cone in  $\mathbb{R}^{n-1}$ . The integral vectors in  $C_{\mathbb{R}}(\sigma_d)$  correspond to the  $\sigma_d$ -semistable dimension vectors. For  $d = \delta$ , since there exists a full exceptional sequence  $(M_{n-2}, \dots, M_2, M_1)$  of  $\sigma_\delta$ -stable representations and since the dimension vectors in an exceptional sequence are linearly independent, we see that  $C_{\mathbb{R}}(\sigma_\delta)$  is a cone of full dimension in  $\mathbb{R}^{n-1}$ . In general, it is well known that  $C_{\mathbb{R}}(\sigma_d)$  is a

cone over a polyhedron where the indivisible dimension vectors in the extremal rays are  $\sigma_d$ -stable dimension vectors. On the other hand, a  $\sigma_d$ -stable dimension vector needs not lie on an extremal ray.

**Lemma 5.1.** *If  $f$  is  $\sigma_d$ -stable and a real Schur root, then it lies in an extremal ray.*

*Proof.* Let  $f$  be a  $\sigma_d$ -stable real Schur root. Assume that  $f$  does not lie on an extremal ray. According to [5, Theorem 6.4], there exists dimension vectors  $f_1, \dots, f_s$ , all lying on extremal rays, such that  $f$  is a positive integral combination of  $f_1, \dots, f_s$  and  $f_1, \dots, f_s$  are linearly independent in  $\mathbb{R}^n$ . Moreover, we have  $\langle f, f_i \rangle \leq 0$  and  $\langle f_i, f \rangle \leq 0$  for all  $1 \leq i \leq s$ . These conditions imply that

$$1 = \langle f, f \rangle = \langle f, a_1 f_1 + \dots + a_s f_s \rangle \leq 0,$$

a contradiction.  $\square$

There is a special case of interest, which is when there is some ray  $[r]$  such that all  $\sigma_d$ -stable dimension vectors, except possibly the ones on  $[s]$ , lie on extremal rays (this is the case when  $d = \delta$  where  $\delta$  is an isotropic Schur root). In such a case, either all  $\sigma_d$ -stable dimension vectors lie on the boundary of the cone  $C_{\mathbb{R}}(\sigma_d)$  or else  $[s]$  lies in the interior of  $C_{\mathbb{R}}(\sigma_d)$ .

Let  $v_1, v_2, \dots, v_r$  be the extremal rays of  $C_{\mathbb{R}}(\sigma_d)$ . Take  $f$  any dimension vector lying in  $C_{\mathbb{R}}(\sigma_d)$  but that is neither in an extremal ray nor in the ray  $[s]$ . Then we know that  $f$  has at least one  $\sigma_d$ -stable factor that lies on an extremal ray. Since the  $\sigma_d$ -stable factors of  $f$  can be ordered to form an orthogonal sequence of Schur roots, we see that there exists a Schur root  $\alpha$  (that corresponds to an extremal ray  $v_i$ ) such that either  $\langle v_i, f \rangle > 0$  or  $\langle f, v_i \rangle > 0$ . If  $\langle v_i, f \rangle > 0$ , then  $\langle v_i, x \rangle = 0$  defines a hyperplane cutting  $C_{\mathbb{R}}(\sigma_d)$  such that  $f, v_i$  lie on the same side while all other extremal rays  $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_r$  of  $C_{\mathbb{R}}(\sigma_d)$  lie on the other side or on the boundary of that hyperplane. We get a similar situation if  $\langle f, v_i \rangle > 0$  by considering the hyperplane  $\langle x, v_i \rangle = 0$ .

For  $1 \leq i \leq r$ , let  $C_i(\sigma_d)$  be the cone in  $\mathbb{R}^{n-1}$  generated by all the rays  $v_1, \dots, v_r$  but  $v_i$ . By the above observation, we have that

$$\text{Proper}(C_{\mathbb{R}}(\sigma_d)) := \bigcap_{1 \leq i \leq r} C_i(\sigma_d)$$

is either empty or else contains only the ray  $[s]$ . We will see that this restriction yield a very beautiful description of  $C_{\mathbb{R}}(\sigma_d)$  and, in particular, of  $C_{\mathbb{R}}(\sigma_\delta)$ .

**Proposition 5.2.** *We have that  $\text{Proper}(C_{\mathbb{R}}(\sigma_d)) = \emptyset$  if and only if  $C_{\mathbb{R}}(\sigma_d)$  is the cone over a simplex.*

*Proof.* The sufficiency is easy to see. Assume that  $\text{Proper}(C_{\mathbb{R}}(\sigma_d)) = \emptyset$ . We may work in  $\mathbb{R}^j$  where  $j \leq n-1$  and assume that  $C_{\mathbb{R}}(\sigma_d)$  is of full dimension in  $\mathbb{R}^j$ . By Radon's theorem, if the number of extremal rays  $r$  of  $C_{\mathbb{R}}(\sigma_d)$  is at least  $(j-1)+2 = j+1$ , then we can partition the rays  $v_1, \dots, v_r$  into two non-empty subsets  $A, B$  such that the corresponding cones  $C(A)$  and  $C(B)$  generated by the rays in  $A$  and by the rays in  $B$  have a ray of intersection. This ray of intersection will have to be in  $\text{Proper}(C_{\mathbb{R}}(\sigma_d)) = \emptyset$ , a contradiction. Therefore,  $r \leq j$ . Since  $C_{\mathbb{R}}(\sigma_d)$  is of full dimension in  $\mathbb{R}^j$ , then  $r = j$  and  $C_{\mathbb{R}}(\sigma_d)$  is the cone over an  $(j-1)$ -simplex.  $\square$

Now, we are interested in the case where  $\text{Proper}(C_{\mathbb{R}}(\sigma_d))$  is reduced to a single ray  $[s]$  (which then has to be the ray of  $\bar{\delta}$  if  $d = \delta$ ). Let us take an affine slice  $\Delta$  of  $C_{\mathbb{R}}(\sigma_d)$ . The rays  $v_1, \dots, v_r$  will correspond to points  $u_1, \dots, u_r$  in  $\Delta$  and these points are the vertices of a polyhedron  $\mathcal{R}$  in  $\Delta$  defined as the convex hull of  $u_1, \dots, u_r$ . The ray  $[s]$  corresponds to a point  $s$  in  $\mathcal{R}$ . In order to study the convex properties of  $\mathcal{R}$ , let us translate  $\mathcal{R}$  so that  $s$  coincides with the origin. In other words, set  $w_i = u_i - s$  and consider the polyhedron  $\mathcal{P}$  which is the convex hull of  $w_1, \dots, w_r$ . Since  $\mathcal{R}$  lies on an affine slice, we see that  $\mathcal{P}$  lies in a subspace of dimension  $n - 2$  of  $\mathbb{R}^{n-1}$ . Let  $\mathcal{P}_i$  be the convex hull of all points  $w_1, \dots, w_r$  but  $w_i$ . We define

$$\text{Proper}(\mathcal{P}) := \cap_{1 \leq i \leq r} \mathcal{P}_i$$

and we will be interested in the case where  $\text{Proper}(\mathcal{P})$  only contains the origin.

The first two lemmas are easy to prove.

**Lemma 5.3.** *Let  $\mathcal{P}'$  be the convex hull of a subset of  $w_1, \dots, w_r$ . Then  $\text{Proper}(\mathcal{P}')$  is either empty or reduced to the origin.*

**Lemma 5.4.** *Consider a non-trivial partition  $\{w_{i_1}, \dots, w_{i_s}\} = A_1 \cup A_2$  of a subset of  $\{w_1, \dots, w_r\}$ . Denote by  $\mathcal{P}_{A_i}$  the convex hull of the points in  $A_i$ , for  $i = 1, 2$ . Then  $\mathcal{P}_{A_1} \cap \mathcal{P}_{A_2}$  is either empty or reduced to the origin.*

**Proposition 5.5.** *Suppose that  $\text{Proper}(\mathcal{P})$  is empty or reduced to the origin and is full dimensional in  $\mathbb{R}^t$  with  $t \leq n - 2$ . Then there exists a vector space decomposition*

$$\mathbb{R}^t = V_1 \oplus \dots \oplus V_s$$

*of  $\mathbb{R}^t$  such that if  $V_i$  has dimension  $d_i$ , then it contains  $d_i + 1$  points among  $0, w_1, \dots, w_r$  that form a  $d_i$ -simplex in  $V_i$  containing the origin.*

*Proof.* We may assume that  $t = n - 2$  so that  $\mathcal{P}$  is  $(n - 2)$ -dimensional. If  $\text{Proper}(\mathcal{P})$  is empty, then  $s = 1$  and the result follows from Proposition 5.2. Assume that  $\text{Proper}(\mathcal{P})$  is reduced to the origin. Suppose first that the origin lies on a facet, say  $F$ , of  $\mathcal{P}$ . We claim that  $F$  contains  $r - 1$  of the points  $w_1, \dots, w_r$ . Assume otherwise. Consider an  $(n - 3)$ -simplex in  $F$  generated by points  $w_{i_1}, \dots, w_{i_{n-2}}$ . Let  $u, v \in \{w_1, \dots, w_r\}$  be two distinct points not in  $F$ . By Radon's theorem, we can partition the points  $\{w_{i_1}, \dots, w_{i_{n-2}}, u, v\}$  into two non-empty subsets  $A_1, A_2$  such that  $\mathcal{P}_{A_1} \cap \mathcal{P}_{A_2} \neq \emptyset$ , where  $\mathcal{P}_{A_i}$  denotes the convex hull of the points in  $A_i$ . By Lemma 5.4, this intersection is the origin and hence lies on  $F$ . Since  $F$  is a facet,  $u, v$  lie on the same side of  $F$ . Therefore, for  $i = 1, 2$ , the set  $B_i := A_i \setminus \{u, v\}$  is not empty. Now,  $B_1$  and  $B_2$  form a partition of  $w_{i_1}, \dots, w_{i_{n-2}}$  such that  $\mathcal{P}_{B_1} \cap \mathcal{P}_{B_2} \neq \emptyset$ , where  $\mathcal{P}_{B_i}$  denotes the convex hull of the points in  $B_i$ . This contradicts Proposition 5.2.

Now, let us assume that the origin lies in the interior of  $\mathcal{P}$ . By Radon's theorem, we can write  $\{w_1, \dots, w_r\} = E_1 \cup E_2$  where  $E_1, E_2$  are disjoint and non-empty such that  $\mathcal{P}_{E_1} \cap \mathcal{P}_{E_2} = \{0\}$ , where  $\mathcal{P}_{E_i}$  denotes the convex hull of the points in  $E_i$ . By Carathéodory's theorem, there is a simplex formed by some points  $z_1, \dots, z_s$  in  $E_1$  that contains the origin in its interior. With no loss of generality, assume that  $z_i = w_i$  and  $s \leq r - 1$ . Let  $V_1$  be the vector space spanned by the points  $w_1, \dots, w_s$  and consider the vector space  $V_2$  spanned by the points  $w_{s+1}, \dots, w_r$ . Let  $C_1$  be the convex hull of the points  $w_1, \dots, w_s$  and let  $C_2$  be the convex hull of the points  $w_{s+1}, \dots, w_r$ . Since both  $\mathcal{P}_{E_1}, \mathcal{P}_{E_2}$  contain the origin, we see that  $C_1 \cap C_2 = \{0\}$ .

Since 0 lies in the interior of  $C_1$ , we get also that  $V_1 \cap C_2 = 0$ . We claim that  $V_1 \cap V_2 = 0$ . Assume that  $V_1 \cap V_2$  is non-zero. Observe that any element in  $V_1$  can be written as a non-negative linear combination of  $w_1, \dots, w_s$ . There exists non-negative real numbers  $a_1, \dots, a_s$  and real numbers  $b_{s+1}, \dots, b_r$  such that

$$a_1 w_1 + \dots + a_s w_s = b_{s+1} w_{s+1} + \dots + b_r w_r.$$

Moreover,  $a_1 w_1 + \dots + a_s w_s$  is non-zero. We may assume the  $a_i$  small enough so that the left-hand side lies in  $C_1$ . Let us write  $\{s+1, \dots, r\} = I_1 \cup I_2$  where  $I_1, I_2$  are disjoint and  $i \in I_1$  if and only if  $b_i \geq 0$ . We may assume further that the  $|b_i|$  are small enough so that both  $\sum_{i \in I_1} b_i w_i, -\sum_{j \in I_2} b_j w_j$  lie in  $C_2$ . If  $I_2 = \emptyset$ , then  $a_1 w_1 + \dots + a_s w_s \in C_1 \cap C_2$  is non-zero, a contradiction. If all  $b_i$  are non-positive, then  $-\sum_{j \in I_2} b_j w_j$  lies in  $V_1 \cap C_2 = \{0\}$ , a contradiction. If some  $b_i$  are negative and some  $b_i$  are positive, we can rewrite the sum as

$$a_1 w_1 + \dots + a_s w_s + -\sum_{j \in I_2} b_j w_j = \sum_{j \in I_1} b_j w_j.$$

Considering Lemma 5.4 with the partition

$$(\{w_1, \dots, w_s\} \cup \{w_i \mid i \in I_2\}) \cup (\{w_j \mid j \in I_1\})$$

of  $\{w_1, \dots, w_r\}$ , we get that  $a_1 w_1 + \dots + a_s w_s + -\sum_{j \in I_2} b_j w_j$  is zero, which reduces to a case we have already considered. Therefore, we have proven that  $\mathbb{R}^{n-2} = V_1 \oplus V_2$ , where  $V_1$  satisfies the property of the statement. We proceed by induction on  $V_2$  with the points  $w_{s+1}, \dots, w_r$  and by using Lemma 5.3.  $\square$

## 6. THE RING OF SEMI-INVARIANTS OF AN ISOTROPIC SCHUR ROOT

In this section, we denote by  $\delta$  an isotropic Schur root and by  $\sigma = \sigma_\delta$  the weight given by  $-\langle -, \delta \rangle$ .

Consider, as previously, a full exceptional sequence  $(M_{n-2}, \dots, M_2, M_1, V, W)$  where  $(M_{n-2}, \dots, M_2, M_1)$  is an exceptional sequence of simple objects in  $\mathcal{A}(\delta)$ . Take  $I \subseteq \{1, \dots, n-2\}$  such that  $i \in I$  if and only if  $\mathcal{C}_i$  is tame connected.

**Definition 6.1.** The associated tame subcategory of  $Q$  relative to  $\delta$ , denoted  $\mathcal{R}(Q, \delta)$ , is the thick subcategory of  $\text{rep}(Q)$  generated by  $(\bigoplus_{i \in I} M_i) \oplus V_{n-2} \oplus W_{n-2}$ .

**Theorem 6.2.** Let  $Q$  be an acyclic connected quiver and  $\delta$  an isotropic Schur root. Then

- (1) The category  $\mathcal{R}(Q, \delta)$  is tame connected with isotropic Schur root  $\bar{\delta}$  and is uniquely determined by  $\delta$ .
- (2) The simple objects in  $\mathcal{A}(\delta)$ , up to isomorphism, are given by the disjoint union

$$\{M_i \mid i \notin I\} \cup \{\text{quasi-simple objects in } \mathcal{R}(Q, \delta)\}.$$

- (3) We have

$$\text{SI}(Q, \delta) \cong \text{SI}(\mathcal{R}(Q, \delta), \bar{\delta})[x_{r+1}, \dots, x_n].$$

*Proof.* Let  $(M_{n-2}, \dots, M_1)$  be an exceptional sequence of  $\sigma_\delta$ -stable representations with the corresponding full exceptional sequence  $(M_{n-2}, \dots, M_1, V, W)$  in  $\text{rep}(Q)$ , where  $\delta = d_V + d_W$ . First, denote by  $M_{l_r}, \dots, M_{l_1}$  with  $l_r > \dots > l_1$  the  $M_j$  such that  $\mathcal{C}_j$  is tame disconnected or such that  $M_j$  is preprojective in  $\mathcal{C}_j$ . We get an exceptional sequence

$$(*) \quad (N_t, \dots, N_2, N_1, V, W, M'_{l_r}, \dots, M'_{l_1})$$

where all  $M_{l_j}$  have been reflected, one by one, to the right of the exceptional sequence. Observe that  $\{M_i \mid i \in I\} \subseteq \{N_1, \dots, N_t\}$ . Let

$$\{N_t, \dots, N_1\} \setminus \{M_i \mid i \in I\} := \{N_{j_s}, \dots, N_{j_1}\}$$

where  $j_s > \dots > j_1$ . Assume also that  $I = \{m_1, \dots, m_q\}$  with  $m_q > \dots > m_1$ . We have  $q + s = t$  and  $t + r = n - 2$ . By Lemma 4.13(2), we may reflect all exceptional objects of  $\{N_{j_s}, \dots, N_{j_1}\}$  in  $(*)$  so that we get an exceptional sequence

$$(M_{m_q}, \dots, M_{m_1}, N_{j_s}, \dots, N_{j_1}, V, W, M'_{l_r}, \dots, M'_{l_1}).$$

Now, it follows from the definition of the  $V_i, W_i$  that we get an exceptional sequence

$$(M_{m_q}, \dots, M_{m_1}, V_{n-2}, W_{n-2}, N_{j_s}, \dots, N_{j_1}, M'_{l_1}, \dots, M'_{l_r}).$$

We claim that for  $1 \leq u \leq q$ , we have that  $\mathcal{C}_{m_u} = \mathcal{C}(M_{m_u}, V_{m_u}, W_{m_u})$  is equivalent to  $\mathcal{C}(M_{m_u}, V_{n-2}, W_{n-2})$ . Fix such a  $u$ . Note that there is an exceptional sequence of the form

$$(N_{j_s}, \dots, N_{j_p}, M_{m_u}, V_{m_u}, W_{m_u}).$$

Now, it follows from Lemma 4.13(2) that  $M_{m_u}$  lies in  ${}^\perp N_{j_i}$  for all  $1 \leq i \leq p$ . By reflecting, we get the exceptional sequence

$$(M_{m_u}, V_{n-2}, W_{n-2}, N_{j_s}, \dots, N_{j_p}).$$

It follows that  $\mathcal{C}_{m_u} = \mathcal{C}(M_{m_u}, V_{m_u}, W_{m_u})$  is equivalent to  $\mathcal{C}(M_{m_u}, V_{n-2}, W_{n-2})$ . This proves our claim. Let  $E = (M_{m_q}, \dots, M_{m_1}, V_{n-2}, W_{n-2})$ . By Lemma 4.12 and our claim,  $\mathcal{R}(Q, \delta) = \mathcal{C}(E)$  is tame connected. Since  $V_{n-2}, W_{n-2}$  lie in it,  $\bar{\delta}$  is the (unique) isotropic Schur root of  $\mathcal{R}(Q, \delta)$ . It follows from Lemma 4.13 that any  $\sigma_\delta$ -stable representation not isomorphic to any  $M_i$  for  $1 \leq i \leq n - 2$  will have to be (quasi-simple) in  $\mathcal{C}(E)$ . Now, we need to show that all quasi-simple objects of  $\mathcal{C}(E)$  are  $\sigma_\delta$ -stable. Assume the contrary. Let  $f$  be the dimension vector of a quasi-simple object in  $\mathcal{C}(E)$  that is not  $\sigma_\delta$ -stable, but  $\sigma_\delta$ -semistable. It follows from our previous observations that  $f$  has to be a positive integral combination of the  $\sigma_\delta$ -stable dimension vectors in  $\mathcal{C}(E)$ . It follows from [11, Cor. 6.3] that this is not possible to have such a decomposition. Therefore, we have a complete list of the simple objects in  $\mathcal{A}(\delta)$ . These are given by the disjoint union

$$\{M_i \mid i \notin I\} \cup \{\text{quasi-simple objects in } \mathcal{R}(Q, \delta)\}.$$

Observe that, in  $C_{\mathbb{R}}(\sigma_\delta)$ , a dimension vector  $d$  can be uniquely written as

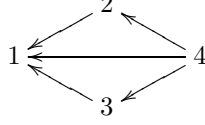
$$d = d_1 + \sum_{i \notin I} \lambda_i f_i$$

where  $d_1$  is a dimension vector in  $\mathcal{C}(E)$  and  $f_i = d_{M_{m_i}}$  for  $i \notin I$ . This decomposition is unique. This implies the unicity of  $\mathcal{R}(Q, \delta)$  and statement (3).  $\square$

**Corollary 6.3.** *Let  $Q$  be an acyclic connected quiver and  $\delta$  an isotropic Schur root. Then  $\text{SI}(Q, \delta)$  is a polynomial ring or a hypersurface. More precisely, it is a hypersurface (and not a polynomial ring) if and only if  $\mathcal{R}(Q, \delta)$  has quiver of type  $\widetilde{\mathbb{D}}_n$  with  $n \geq 4$ ,  $\widetilde{\mathbb{E}}_6$ ,  $\widetilde{\mathbb{E}}_7$  or  $\widetilde{\mathbb{E}}_8$ .*

*Proof.* In [20], it was proven that the ring of semi-invariant of an isotropic Schur root of a tame quiver is a polynomial ring or a hypersurface, where the second situation occurs precisely when the quiver is of type  $\widetilde{\mathbb{D}}_n$  with  $n \geq 4$ ,  $\widetilde{\mathbb{E}}_6$ ,  $\widetilde{\mathbb{E}}_7$  or  $\widetilde{\mathbb{E}}_8$ . Our result follows from this and Theorem 6.2.  $\square$

**Example 6.4.** Consider the quiver  $Q$  given by



Consider the exceptional sequence  $(P_2, S_1, I_3, S_3)$  where  $P_2$  is the projective representation at vertex 2,  $I_3$  is the injective representation at vertex 3 and  $S_1, S_3$  are the simple representations at vertices 1, 3, respectively. Reflecting  $S_1, I_3$  to the left of  $P_2$ , we get an exceptional sequence whose dimension vectors are as follows.

$$((0, 1, 0, 0), (3, 3, 1, 1), (1, 1, 0, 0), (0, 0, 1, 0)).$$

Then, using a sequence of reflections, we get the following exceptional sequences, where we put the corresponding dimension vectors.

$$\begin{aligned} &((0, 1, 0, 0), (3, 3, 1, 1), (0, 0, 1, 0), (1, 1, 1, 0)) \\ &((0, 1, 0, 0), (0, 0, 1, 0), (3, 3, 3, 1), (1, 1, 1, 0)) \\ &((0, 0, 1, 0), (0, 1, 0, 0), (3, 3, 3, 1), (1, 1, 1, 0)) \\ &((0, 0, 1, 0), (0, 1, 0, 0), (8, 8, 8, 3), (3, 3, 3, 1)) \\ &((0, 0, 1, 0), (8, 3, 8, 3), (0, 1, 0, 0), (3, 3, 3, 1)) \\ &((8, 3, 3, 3), (0, 0, 1, 0), (0, 1, 0, 0), (3, 3, 3, 1)). \end{aligned}$$

Observe that

$$\langle (3, 3, 3, 1), (0, 1, 0, 0) \rangle = 2$$

and

$$\delta = (3, 3, 3, 1) - (0, 1, 0, 0) = (3, 2, 3, 1)$$

is an isotropic Schur root. The Coxeter matrix  $\tau$  is

$$\tau = \begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ -1 & 1 & 0 & 2 \\ -3 & 2 & 2 & 4 \end{pmatrix}.$$

This matrix has eigenvalues  $\lambda = 5/2 + \sqrt{21}/2$ ,  $\lambda^{-1} = 5/2 - \sqrt{21}/2$  and  $-1$  with (algebraic and geometric) multiplicity 2. The eigenvector corresponding to  $\lambda$  is

$$v_1 = (10, 9 + \sqrt{21}, 9 + \sqrt{21}, 17 + \sqrt{189})$$

and the one corresponding to  $\lambda^{-1}$  is

$$v_2 = (10, 9 - \sqrt{21}, 9 - \sqrt{21}, 17 - \sqrt{189}).$$

Now,  $\langle v_2, (8, 3, 3, 3) \rangle = -197 + 10\sqrt{21} + 11\sqrt{189} > 0$  and  $\langle v_2, (0, 0, 1, 0) \rangle = -8 - \sqrt{21} + \sqrt{189} > 0$ . Similarly, both  $\langle (8, 3, 3, 3), v_1 \rangle$  and  $\langle (0, 0, 1, 0), v_1 \rangle$  are positive. Therefore, the exceptional objects with dimension vectors  $(8, 3, 3, 3), (0, 0, 1, 0)$  are regular by the theorem at page 240 of [16]. It follows from Proposition 4.10 that  $\delta$  is not of smaller type. It also follows from the same proposition that there is a unique exceptional sequence  $(M_2, M_1)$  of length 2 of  $\sigma$ -stable objects. Let  $M'_1 = S_3$  and  $M'_2$  be the exceptional representation with dimension vector  $(8, 3, 3, 3)$ . Since  $M'_1, M'_2$  lie in  $\mathcal{C}(M_2, M_1)$  by Lemma 3.8, we see that  $\mathcal{C}(M'_2, M'_1) \subseteq \mathcal{C}(M_2, M_1)$  and thus, we have equality. This means that  $M'_2 = M_2$ ,  $M'_1 = M_1$ . Since  $\langle \delta, (0, 0, 1, 0) \rangle = 2 > 0$ , we get  $\delta_1 = \delta - 2(0, 0, 1, 0) = (3, 2, 1, 1)$ . Now,  $\langle \delta_1, (8, 3, 3, 3) \rangle = -2$ . Therefore,



$\bar{\delta} = \delta_1 = (3, 2, 1, 1)$ . In this example, the cone of  $\sigma$ -semistable dimension vectors is as follows (where only an affine slice of that cone is shown).

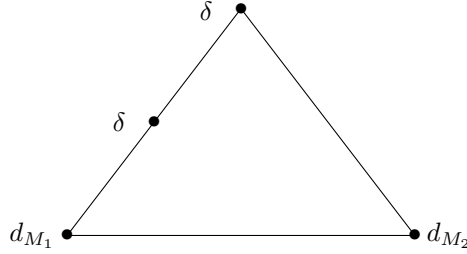


FIGURE 1. The cone of  $\sigma$ -semistable dimension vectors for  $\delta = (3, 2, 3, 1)$

The following is an easy observation. The reader is referred to [8] for the notion of cluster algebra and to [7] for results in similar directions.

**Corollary 6.5.** *If  $\text{SI}(Q, \delta)$  is not a polynomial ring, then it has a cluster algebra structure of type  $\mathbb{A}_1$ . There are two cluster variables which are all  $\Gamma$ -homogeneous, and the coefficients are built from  $n-1$  frozen variables, which are also  $\Gamma$ -homogeneous, where  $\Gamma$  is the set of all multiplicative characters of  $\text{GL}_\delta(k)$ .*

*Proof.* From Theorem 6.2, it is enough to prove this for  $\text{rep}(Q) = \mathcal{R}(Q, \delta)$ , that is, we may assume that  $Q$  is tame connected. Suppose that  $\text{SI}(Q, \delta)$  is not a polynomial ring. Then  $Q$  is of type  $\widetilde{\mathbb{D}}_n$  with  $n \geq 4$ ,  $\widetilde{\mathbb{E}}_6$ ,  $\widetilde{\mathbb{E}}_7$  or  $\widetilde{\mathbb{E}}_8$ . In particular, it is well known in these cases that there are exactly three non-homogeneous tubes  $T_1, T_2, T_3$  in the Auslander-Reiten quiver of  $\mathcal{R}(Q, \delta)$ . One, say  $T_1$ , has rank 2. Let  $M, N$  be the non-isomorphic exceptional quasi-simple objects in  $T_1$ . Then, let  $E_1, \dots, E_r$  be the non-isomorphic quasi-simple objects of  $T_2$  and let  $E'_1, \dots, E'_t$  be the non-isomorphic quasi-simple objects of  $T_3$ . Now, the hypersurface equation can be written as

$$(*) \quad C^M C^N = C^{E_1} \dots C^{E_r} + C^{E'_1} \dots C^{E'_t}.$$

Consider the indeterminates  $x, y_1, \dots, y_r, z_1, \dots, z_t$ . We define a cluster algebra  $A$  as follows. We start with the initial seed  $\{x, y_1, \dots, y_r, z_1, \dots, z_t\}$  where  $y_1, \dots, y_r$  and  $z_1, \dots, z_t$  are declared to be frozen variables. The exchange relation is  $xx' = \prod_{i=1}^r y_i + \prod_{j=1}^t z_j$  which clearly produces exactly two cluster variables  $x, x'$ . The cluster algebra is the  $\mathbb{Z}$ -subalgebra of  $\mathbb{Q}(x, y_1, \dots, y_r, z_1, \dots, z_t)$  generated by  $x, x'$  and  $y_1, \dots, y_r, z_1, \dots, z_t$ . This algebra is clearly isomorphic to  $\text{SI}(Q, \delta)$ .  $\square$

An interesting problem would be to find all acyclic quivers  $Q$  and dimension vectors  $d$  such that  $\text{SI}(Q, d)$  has a cluster algebra structure whose variables (frozen or not) are all  $\Gamma$ -homogeneous.

## 7. CONSTRUCTION OF ALL ISOTROPIC SCHUR ROOTS

In this section, we show that all of the isotropic Schur roots of  $\text{rep}(Q)$  come from isotropic Schur roots of a tame full subquiver of  $Q$  by applying special reflections. We make this precise by defining an action of the braid group  $B_{n-1}$  on  $n-1$  strands on a special type of exceptional sequences that will encode all we need to study isotropic Schur roots. We start with the definition of these sequences.

**Definition 7.1.** Let  $E = (X_1, \dots, X_n)$  be a full exceptional sequence. We say that  $E$  is of *isotropic type* if there exists  $1 \leq i \leq n-1$  such that  $\mathcal{C}(X_i, X_{i+1})$  is tame. The integer  $i$  is called the *isotropic position* of  $E$  and the *root type* of  $E$ , denoted  $\delta_E$ , is the isotropic Schur root in  $\mathcal{C}(X_i, X_{i+1})$ .

We denote by  $\mathcal{E}$  the set of all full exceptional sequences of isotropic type, up to isomorphism. Not all elements of the braid group  $B_n$  act on  $\mathcal{E}$ . We rather consider the group  $B_{n-1}$  and show that it acts on  $\mathcal{E}$ . Let us denote the standard generators of  $B_{n-1}$  by  $\gamma_1, \dots, \gamma_{n-2}$ . Let  $E = (X_1, \dots, X_n) \in \mathcal{E}$  with isotropic position  $r$ . Let  $1 \leq i \leq n-2$ . If  $i < r-1$ , then  $\gamma_i E := \sigma_i E$ . If  $i > r$ , then  $\gamma_i E := \sigma_{i+1} E$ . Assume that  $i = r$  with  $r < n-1$ . We can reflect  $X_{r+2}$  to the left of  $X_r, X_{r+1}$  to get the exceptional sequence:

$$E' = (X_1, \dots, X_{r-1}, L_{X_r}(L_{X_{r+1}}(X_{r+2})), X_r, X_{r+1}, X_{r+3}, \dots, X_n).$$

and this is an exceptional sequence of isotropic type with isotropic position  $r+1$ . We define  $\gamma_r E := E'$ . If  $r > 1$  and  $i = r-1$ , then we can reflect both  $X_r, X_{r+1}$  to the left of  $X_{r-1}$  as follows:

$$E'' = (X_1, \dots, X_{r-2}, L_{X_{r-1}}(X_r), L_{X_{r-1}}(X_{r+1}), X_{r-1}, X_{r+2}, \dots, X_n)$$

and clearly, the subcategory  $\mathcal{C}(L_{X_{i-1}}(X_i), L_{X_{i-1}}(X_{i+1}))$  generates a tame subcategory of rank 2. Therefore,  $E''$  is an exceptional sequence of isotropic type with isotropic position  $r-1$  and its root type is the unique isotropic Schur root in  $\mathcal{C}(L_{X_{i-1}}(X_i), L_{X_{i-1}}(X_{i+1}))$ , which is  $\delta_{E'} = \delta_E - \langle \delta_E, d_{X_{r-1}} \rangle d_{X_{r-1}}$ , by Lemma 4.5. We define  $\gamma_{r-1} E = E''$ . Similarly, we can define the action of  $\gamma_i^{-1}$  on  $E$  for  $1 \leq i \leq n-2$ . The following is easy to check.

**Proposition 7.2.** *The group  $B_{n-1}$  acts on exceptional sequences of isotropic type, with the action defined above.*

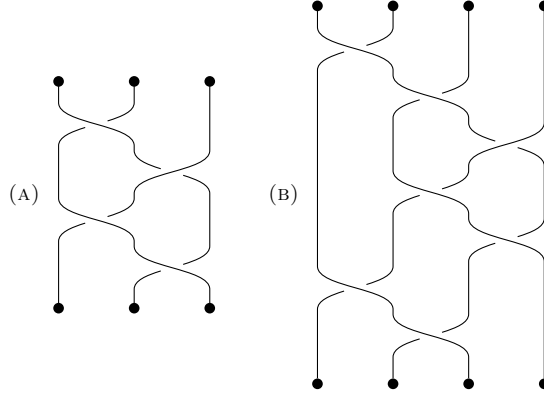
**Definition 7.3.** A sequence  $E = (X_1, \dots, X_{n-1}, X_n)$  in  $\mathcal{E}$  is of *tame type* if it has isotropic position  $n-1$ , and there is  $0 \leq s \leq n-2$  such that  $X_1, \dots, X_s$  are projective in  $\text{rep}(Q)$  and  $\mathcal{C}(X_{s+1}, \dots, X_{n-2}, X_{n-1}, X_n)$  is tame connected. By convention,  $s = 0$  means that  $\text{rep}(Q)$  is already tame connected.

Observe that if  $E \in \mathcal{E}$  is of tame type, then the isotropic Schur root  $\delta_E$  is the unique isotropic Schur root of the tame subcategory  $\mathcal{C}(X_{s+1}, \dots, X_{n-2}, X_{n-1}, X_n)$  and is an isotropic Schur root coming from a tame full subquiver of  $Q$ . In particular, there are finitely many roots  $\delta_E$  where  $E \in \mathcal{E}$  is of tame type.

**Example 7.4.** Consider a quiver of rank  $n = 4$  and an exceptional sequence  $E = (X, U, V, Y)$  of isotropic type with isotropic position 2. The root type is the isotropic root  $\delta_E$  in  $\mathcal{C}(U, V)$ .

The first braid (A) in Figure 2 corresponds to the element  $g = \gamma_2^{-1} \gamma_1^{-1} \gamma_2 \gamma_1^{-1}$  of  $B_3$  while the second braid (B) corresponds to the element  $h = \sigma_2^{-1} \sigma_1^{-1} \sigma_3^{-1} \sigma_2^{-1} \sigma_3 \sigma_2^{-1} \sigma_1^{-1}$  of  $B_4$ . Notice that  $gE = hE$ . Notice also that the braid in (A) is obtained from the braid in (B) by identifying the two strands starting at the positions of  $U, V$ , that is, the second and third strands.

Our aim in this section is to prove that any  $E \in \mathcal{E}$  lies in the  $B_{n-1}$ -orbit of an exceptional sequence of tame type. In the next lemmas, we will consider exceptional sequences in the bounded derived category  $D^b(\text{rep}(Q))$  of  $\text{rep}(Q)$ . Recall that an object  $X$  in  $D^b(\text{rep}(Q))$  is *exceptional* if  $\text{Hom}(X, X[i]) = 0$  for all non-zero

FIGURE 2. Correspondence between  $B_3$  and some braids of  $B_4$ 

$i$  (and then,  $\text{Hom}(X, X)$  has to be one dimensional). Equivalently, an exceptional object in  $D^b(\text{rep}(Q))$  is isomorphic to the shift of an exceptional representation. A sequence  $(X_1, \dots, X_r)$  of objects in  $D^b(\text{rep}(Q))$  is *exceptional* if every  $X_i$  is exceptional and, for  $i < j$ , we have  $\text{Hom}_{D^b(\text{rep}(Q))}(X_i, X_j[t]) = 0$  for all  $t \in \mathbb{Z}$ . For such a sequence, one can consider the smallest full additive subcategory  $\mathcal{D}(X_1, \dots, X_r)$  of  $D^b(\text{rep}(Q))$  containing  $X_1, \dots, X_r$  and that is closed under direct sums, direct summands, taking the cone of a morphism and the shift of an object. One can also consider the exceptional sequence  $(X'_1, \dots, X'_r)$  in  $\text{rep}(Q)$  such that  $X'_i$  is the unique shift of  $X_i$  lying in  $\text{rep}(Q)$ . The indecomposable objects in  $\mathcal{D}(X_1, \dots, X_r)$  are just the shifts of the indecomposable objects in  $\mathcal{C}(X'_1, \dots, X'_r)$ .

In what follows, the Auslander-Reiten translate in  $D^b(\text{rep}(Q))$  is denoted by  $\tau_D$  while the Auslander-Reiten translate in  $\text{rep}(Q)$  is simply denoted  $\tau$ . Recall that if  $X$  is a non-projective indecomposable representation, then  $\tau_D X = \tau X$  and, if  $X = P_x$  with  $x \in Q_0$ , then  $\tau_D X = I_x[-1]$ . When  $d$  is a dimension vector, we denote by  $\tau d$  the product of the Coxeter matrix with  $d$ . In particular, if  $X$  is a non-projective indecomposable representation, then  $\tau d_X = d_{\tau X}$  and, if  $X = P_x$  with  $x \in Q_0$ , then  $\tau d_X = -d_{I_x}$ . We start our investigation with the following lemma that is crucial for the proof of the main result of this section.

**Lemma 7.5.** *Let  $(X_1, \dots, X_n)$  be an exceptional sequence with  $\mathcal{C}(X_{r+1}, \dots, X_n)$  tame and assume that  $X_1, \dots, X_r$  are the simple objects in  $\mathcal{C}(X_1, \dots, X_r)$ . Let  $X \in \mathcal{C}(X_1, \dots, X_r)$  be the injective object with socle  $X_1$ . If  $X$  is projective in  $\mathcal{C}(X, X_{r+1}, \dots, X_n)$ , then  $X_1$  is projective in  $\text{rep}(Q)$  and in particular, an isotropic Schur root of  $\mathcal{C}(X_{r+1}, \dots, X_n)$  is not sincere.*

*Proof.* Assume that  $X$  is projective in  $\mathcal{C}(X, X_{r+1}, \dots, X_n)$ . Set  $d_i = d_{X_i}$  for  $1 \leq i \leq n$ . Consider the linear form  $f$  given by  $f(x) = \langle d_1, x \rangle$ . Then  $f$  vanishes on  $d_2, \dots, d_n$  and  $f(d_1) > 0$ . Assume to the contrary that  $X_1$  is not projective in  $\text{rep}(Q)$ . Observe that  $f(x) = \langle d_1, x \rangle = -\langle x, \tau d_1 \rangle$ . Since  $\tau X_1$  is exceptional,  $\langle \tau d_1, \tau d_1 \rangle = 1$  and hence  $f(\tau d_1) < 0$ . Now, reflect  $X_1$  to the right of  $X_2, \dots, X_r$ , so that we get an exceptional sequence  $(X_2, \dots, X_r, Y)$  where  $Y$  is in the cone spanned by  $d_1, \dots, d_r$ . Clearly,  $X_1$  is simple projective in  $\mathcal{C}(X_1, \dots, X_r)$  and hence,  $Y = X$  is the injective hull of  $X_1$  in  $\mathcal{C}(X_1, \dots, X_r)$ . Set  $\mathcal{C} := \mathcal{C}(X, X_{r+1}, \dots, X_n)$ . We know that  $X$  is projective in  $\mathcal{C}$ . Reflecting  $X$  to the right of  $X_{r+1}, \dots, X_n$  will give the

exceptional representation  $\tau X_1$ . Therefore,  $d := \tau d_1 = -\tau_C d_X$  where  $\tau_C$  denotes the Coxeter transformation in  $\mathcal{C}$ . Take the linear form  $g$  in the Grothendieck group of  $\mathcal{C}$  given by  $g(x) = \langle d_X, x \rangle$ . Then  $g$  vanishes on  $d_{r+1}, \dots, d_n$  and  $g(d_X) > 0$ . The form  $f|_{\mathcal{C}}$  has the same property since  $d_X$  is a non-negative linear combination of  $d_1, \dots, d_r$  with the coefficient of  $d_1$  positive. Thus,  $g = f|_{\mathcal{C}}$  up to a positive scalar. Therefore,  $g(d) < 0$ , which means that  $X, \tau X_1$  lie on opposite sides of the hyperplane  $g(x) = 0$  in  $\mathcal{C}$ . This contradicts that  $X$  is projective in  $\mathcal{C}$ .  $\square$

**Lemma 7.6.** *Let  $(X_1, \dots, X_r)$  be an exceptional sequence and assume that  $\mathcal{C}_1 := \mathcal{C}(X_2, \dots, X_r)$  is tame with an isotropic Schur root  $\gamma$  while  $\mathcal{C}_2 := \mathcal{C}(X_1, X_2, \dots, X_r)$  is wild. Then there is a unique minimal isotropic Schur root in the  $\tau$ -orbit of  $\gamma$ .*

*Proof.* We may assume that  $\mathcal{C}_2 = \text{rep}(Q)$  for an acyclic quiver  $Q$ . Since  $\mathcal{C}_2$  is wild and  $\mathcal{C}_1$  is tame, we know that  $X_1$  is preprojective or preinjective in  $\mathcal{C}_2$ . Hence, there is some  $r \in \mathbb{Z}$  such that  $\tau_D^r X_1$  is projective or the shift of a projective. This means that  $\tau^r \gamma$  is not sincere. Let  $Y = \tau_D^r X_1$  if  $\tau_D^r X_1$  is a representation or  $Y = \tau_D^r X_1[-1]$  if  $\tau_D^r X_1$  is the shift of a projective representation. Observe that  $Y^\perp \subseteq \text{rep}(Q)$  is also of tame representation type, where the quivers of  $X_1^\perp$  and  $Y^\perp$  only differ by a change of orientation; see for instance [10, Prop. 2.1]. Therefore,  $\tau^r \gamma$  is an isotropic Schur root of a tame full subquiver of  $Q$ . Let  $s \in \mathbb{Z}$  with  $s \neq r$ . Consider  $Z$  the unique shift of  $\tau_D^s X_1$  which is a representation. Since the simples in  $Y^\perp$  are simples in  $\text{rep}(Q)$ , and since there is a simple of  $Z^\perp \subseteq \text{rep}(Q)$  that is not simple in  $\text{rep}(Q)$ , we see that the isotropic Schur root  $\tau^r \gamma$  has smaller length than  $\tau^s \gamma$ . This also proves unicity since only one object in the  $\tau$ -orbit of  $X_1$  is projective or a shift of a projective.  $\square$

**Lemma 7.7.** *Let  $E = (X_1, \dots, X_{n-2}, U, V)$  be in  $\mathcal{E}$  with isotropic position  $n-1$ . Let  $E' = \gamma_1 \cdots \gamma_{n-3} \gamma_{n-2} E = (U', V', X_1, \dots, X_{n-2})$ . Then  $\tau^{-1} \delta_E = \delta_{E'}$ .*

*Proof.* If  $V$  is not injective, then we have the exceptional sequence

$$(\tau^{-1} V, X_1, \dots, X_{n-2}, U)$$

in  $\text{rep}(Q)$ . Otherwise, we have the exceptional sequence

$$(\tau^{-1} V[-1], X_1, \dots, X_{n-2}, U)$$

in  $\text{rep}(Q)$ . Let us write  $\tau^{-1} V[0, -1]$  to indicate that we either take the shift  $[0]$  or  $[-1]$  for  $\tau^{-1} V$ . Then, we get an exceptional sequence

$$(\tau^{-1} U[0, 1], \tau^{-1} V[0, 1], X_1, \dots, X_{n-2}).$$

The categories  $\mathcal{C}(U', V')$  and  $\mathcal{C}(\tau^{-1} U[0, 1], \tau^{-1} V[0, 1])$  are equal in  $\text{rep}(Q)$ . Therefore, they have the same isotropic Schur root. The isotropic Schur root of

$$\mathcal{C}(\tau^{-1} U[0, 1], \tau^{-1} V[0, 1])$$

is clearly  $\tau^{-1} \delta_E$ .  $\square$

Of course, we have the dual version of the above lemma as follows.

**Lemma 7.8.** *Let  $E = (U, V, X_1, \dots, X_{n-2})$  be in  $\mathcal{E}$  with isotropic position 1. Let  $E' = \gamma_{n-2}^{-1} \gamma_{n-3}^{-1} \cdots \gamma_1^{-1} E = (X_1, \dots, X_{n-2}, U', V')$ . Then  $\tau \delta_E = \delta_{E'}$ .*

We are now ready for the main result of this section.

**Theorem 7.9.** *Let  $\delta$  be an isotropic Schur root. Then there is  $E \in \mathcal{E}$  of tame type and  $g \in B_{n-1}$  such that  $gE$  has root type  $\delta$ .*

*Proof.* It follows from Proposition 4.1 that there is an exceptional sequence  $F = (M_1, \dots, M_{n-2}, X, Y)$  in  $\mathcal{E}$  of isotropic position  $n-1$  and of root type  $\delta$ . Assume that  $G \in \mathcal{E}$  is in the orbit of  $E$  and the root type of  $G$  is minimal, that is, has minimal length as a root in  $\text{rep}(Q)$ . We may assume that the isotropic position of  $G$  is  $n-1$ . Therefore, we may assume that  $G$  is of the form

$$(Y_1, \dots, Y_{n-2}, U, V).$$

Assume first that there is an object  $W$  in  $\mathcal{C}(Y_1, \dots, Y_{n-2})$  that is not projective in  $\mathcal{C}(W, U, V)$ . We can apply a sequence of reflections to the subsequence  $(Y_1, \dots, Y_{n-2})$  to get an exceptional sequence

$$H = (Y'_1, \dots, Y'_{n-3}, W, U, V)$$

in  $\mathcal{E}$ . Now, applying  $\gamma_{n-2}^2$  to  $H$  and using Lemma 7.7, we get the sequence

$$(Y'_1, \dots, Y'_{n-3}, W', U', V')$$

in  $\mathcal{E}$  whose root type is the inverse Auslander-Reiten translate of  $\delta$  in  $\mathcal{C}(W, U, V)$ . Similarly, applying  $(\gamma_{n-2}^{-1})^2$  to  $H$ , we get the sequence

$$(Y'_1, \dots, Y'_{n-3}, W'', U'', V'')$$

in  $\mathcal{E}$  whose root type is the Auslander-Reiten translate of  $\delta$  in  $\mathcal{C}(W, U, V)$ . We can iterate this to get a smaller root by Lemma 7.6, provided  $\mathcal{C}(W, U, V)$  is wild. Therefore, whenever there is an object  $W$  which is not projective in  $\mathcal{C}(W, U, V)$ , then  $\mathcal{C}(W, U, V)$  is of tame type (and hence connected). Suppose, by induction, that we have an exceptional sequence  $J = (W_{r+1}, \dots, W_{n-2}, U, V)$  such that  $\mathcal{C}(J)$  is tame connected and  $J$  has maximal length with respect to this property. If  $r = 0$ , then  $Q$  is a tame connected quiver and there is nothing to prove. Complete this to get a full exceptional sequence

$$(Z_1, \dots, Z_r, W_{r+1}, \dots, W_{n-2}, U, V).$$

If there is  $W \in {}^\perp \mathcal{C}(J) = \mathcal{C}(Z_1, \dots, Z_r)$  such that  $W$  is not projective in  $\mathcal{C}(W, U, V)$  then,  $\mathcal{C}(W, U, V)$  is tame connected. As in the proof of Lemma 4.12, we get that  $\mathcal{C}(W, W_{r+1}, \dots, W_{n-2}, U, V)$  is tame connected, contradicting the maximality of  $J$ . Therefore, any object  $Z$  in  $\mathcal{C}(Z_1, \dots, Z_r)$  is such that  $Z$  is projective in  $\mathcal{C}(Z, W_{r+1}, \dots, W_{n-2}, U, V)$ . We may apply a sequence of reflections and assume that all of  $Z_1, \dots, Z_r$  are simple in  $\mathcal{C}(Z_1, \dots, Z_r)$ . It follows from Lemma 7.5 that the injective hull of  $Z_1$  in  $\mathcal{C}(Z_1, \dots, Z_r)$  is projective in  $\text{rep}(Q)$ . Then the proof goes by induction.  $\square$

Here is another way to interpret this result. Start with an isotropic Schur root  $\delta_0$  of a tame full subquiver  $Q'$  of  $Q$  and consider an exceptional sequence  $(U_0, V_0)$  of length 2 in  $\text{rep}(Q') \subset \text{rep}(Q)$  such that  $\delta_0 = d_{U_0} + d_{V_0}$ . Consider an exceptional object  $X_0$  such that  $(X_0, U_0, V_0)$  is an exceptional sequence of length three (which generates a thick subcategory  $\mathcal{C}_0$  of  $\text{rep}(Q)$ ). Then we can transform it into another exceptional sequence  $(X'_0, U_1, V_1)$  with an isotropic Schur root  $\delta_1 = d_{U_1} + d_{V_1}$  such that  $\delta_1$  is a power  $\tau_{\mathcal{C}_0}^{r_0} \delta_0$  where  $\tau_{\mathcal{C}_0}$  denotes the Coxeter matrix for  $\mathcal{C}_0$ . Now, for  $i \geq 1$ , consider an exceptional object  $X_i$  such that  $(X_i, U_i, V_i)$  is an exceptional sequence. Take a power  $\delta_{i+1} = \tau_{\mathcal{C}_i}^{r_i} \delta_i$  where  $\mathcal{C}_i$  is the thick subcategory of  $\text{rep}(Q)$  generated by  $X_i, U_i, V_i$  and  $\tau_{\mathcal{C}_i}$  denotes the Coxeter matrix for  $\mathcal{C}_i$ . All the roots  $\delta_i$  constructed this way are isotropic Schur roots. Moreover, all isotropic Schur roots of  $\text{rep}(Q)$  can be obtained in this way. There are clearly only finitely many starting

roots  $\delta_0$ , but the choices of the  $r_i$  and  $X_i$  yield, in general, infinitely many possible isotropic Schur roots. As observed in [15], when  $Q$  is wild connected with more than 3 vertices, there are infinitely many  $\tau$ -orbit of isotropic Schur roots (provided there is at least one isotropic Schur root). An interesting question would be to describe the minimal root types of the orbits of  $\mathcal{E}$  under  $B_{n-1}$ . It is not hard to check that when  $n = 3$ , these minimal root types correspond exactly to the tame full subquivers of  $Q$ . We do not know if this holds in general.

**Conjecture 7.10.** *Let  $E_1, E_2 \in \mathcal{E}$ . Assume that there are  $g_1, g_2 \in B_{n-1}$  with  $g_1 E_1, g_2 E_2$  of tame type but with different root types. Then  $E_1, E_2$  lie in distinct orbits under  $B_{n-1}$ .*

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